

PH.D. THESIS

INFORMATIONAL CASCADE AS
A PINNING CONTROL PROBLEM

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To A.

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Introduction

1.1 Motivation

“In the face of the crisis, we felt abandoned by conventional tools... we need to develop complementary tools to improve robustness of our overall framework... I would very much welcome inspiration from other disciplines: physics, engineering, biology. Bringing experts from these fields together with economist and central bankers is potentially very creative and valuable. Scientists have developed sophisticated tools for analyzing complex dynamic systems in rigorous way.”

*Jean-Claude Trichet,
European Central Bank Governor from 2003 to 2011,
in his speech at the ECBs flagship annual Central Banking Conference,
2010.*

Neoclassical economics plays a fundamental role in the study of price and income distributions in markets [8]. However, being focused on the analysis of equilibrium configurations, it may fail into predicting sudden changes in the markets dynamics. Indeed, as pointed out by Lord

Trichet, the recent economic crisis has highlighted the limitations of the existing economic and financial models, which had been incapable of predicting and explaining its driving factors [51]. These considerations led to a tremendous interest in the scientific literature on the development of tools and approaches that might complement neoclassical economics, removing some of its main assumptions, such as rationality and homogeneity of the financial agents [30]. In particular, one of the crucial issues of traditional economics is the complete disregarding of the interactions among the agents. Actually, recent develops in behavioral economics had shown the natural tendency oh human beings of imitate other people, learning from the behavior of others [100]. This tendency is generally known as “herding behavior” and is a very common phenomenon characterizing human life. Since in early childhood, for instance, babies try to imitate the adults around them. They mimic the actions and the facial expressions of adults. This is how they learn about what certain actions signify. In financial markets, this predisposition to imitate is deeply rooted. Investors may abandon what they believe to be their own available *information* on the market and follow the behavior of other investors instead, although their own information indicates that they should have an entirely different behavior. If all the traders behave in the same way, the possible result might be that all, or most of them, take the same investment decision, triggering what is called “informational cascade”. This could lead to substantial price fluctuations in the market, and eventually to a financial crisis. Informational cascades are the evidence that individual rationality may lead to group irrationality.

Herding in financial markets, therefore, may arise when payoffs are similar even if personal information on the market is not. In this case people communicate with each other or observe the actions of others,

or the consequences of these actions. The key issue is how individuals determine which alternative is better. Each individual could decide by direct analysis of the alternatives. However, this can be costly and time-consuming, so a plausible alternative is to rely on the information of others. Such influence may take the form of direct communication and discussion with, or simply observation of others. In any case, it requires a kind of *interaction*. Thus, interactions among the agents, disregarded by neoclassical economy, are crucial to understand the market dynamics.

In order to overcome the limitations of traditional models, a special interest emerged for a complex system approach, which involves the use of agent-based models [84, 25, 30, 50, 5, 26, 74, 75, 83]. In particular, the European Union is supporting the research in this area under the 7th framework program, see for instance the project CRISIS [1]. Agent-based models take advantage of the increased computational capabilities to model financial markets at a microscopical level: they allow to simulate the behavior of a (possibly high) number of decision-makers and institutions, interacting through prescribed rules; the agents may be heterogeneous and have the capability of adapting their actions according to their current situation, the inputs from the environment [80], and the rules governing their behaviour [9].

Complex networks paradigm represents a fundamental tool in this perspective: exploiting the link between multi-agent systems and graph theory [13], each agent can be seen as a node of a dynamical network, and the topology of interconnections well model the reciprocal influence among the agents. This allows to take into account lots of scenarios of interactions and different kinds of agents' behavior, differently from neoclassical models, and eventually to investigate the macro features of the market emerging from the local interactions among the agents.

In this perspective, the purpose of this thesis is to make a further step toward the understanding and the modeling of financial dynamics. We are confident that our scientific community could give its contribution in this direction. In particular, we will focus on the phenomenon of informational cascades occurring in financial markets. Our aim is to provide new tools from *nonlinear dynamical systems* and *control theory*, which, matched with complex networks and agent-based modeling, allow to model, analyze and eventually predict such phenomena.

Mixing tools from agent-based modeling and complex networks, we first propose a reference scenario of artificial financial market, in which each agent is modeled as a nonlinear dynamical system. Different environmental features and agents' behaviors are implemented, with a particular focus on the agents' interactions, in order to observe the possible tendency of the agents of converging towards a general consensus.

The next step is that of analyzing the effects of herd behavior in such scenarios. In the existing models of informational cascades, the presence of an exogenous information available to all or a subset of agents is assumed. In line with this choice, we test the response of our market to exogenous factors, such as a new available information.

Our aim is to treat herding from a new perspective, that is, with a control theory approach. We start by noting that herding is actually a diffusive process, as it generates the spreading of a certain opinion across the financial traders which leads them toward the same trading action. Consequently, we propose a new model of opinion dynamics in which the agents influence each other through a diffusive coupling, in order to capture the tendency of the agents of following the trading actions of some other agents they consider better informed. In this model, the exogenous information becomes an external *signal* exerted on a subset of traders. In the view of this, we can treat herding as a

pinning control problem, so to leverage the contributions in the field of pinning control to predict the emergence of informational cascades. Moreover, differently from the models already present in the literature which only generate total informational cascades [7, 11, 15, 137], we exploit the recent contributions in the field of partial pinning control [37] to take into account the more realistic case of partial cascades, which do not involve all the financial traders, as shown by empirical evidence [70, 127]. We show our ability of modeling and predicting the intensity of such phenomena in artificial markets exploiting tools from control theory and nonlinear dynamical systems, providing a little contribution in the hard task of understanding and explaining financial dynamics. Eventually, we propose an alternative method, based on the generating functions approach [96, 95], to analytically predict information about partial pinning controllability of a given network.

1.2 Thesis Outline

The thesis is organized as follows. In Chapter 2, we will introduce the problem of synchronization of complex networks, with a particular focus on pinning control methods. In Chapter 3, we will build an agent-based model of artificial financial market. In Chapter 4, we define the concepts of herding behavior and informational cascades, focusing on the classical models of informational cascades already present in the literature. In Chapter 5, we will first review the existing models of opinion dynamics. Then, we will introduce a novel opinion dynamics model which accounts for the phenomenon of partial informational cascades. The actual capability of the proposed model of triggering informational cascades of different intensities will be extensively exposed in Chapter 6. Here, we will explain how informational cascades can be seen as a pinning control problem. This will allow to leverage tools of pinning control theory to

predict such phenomena. We will test our model in the artificial financial market proposed in Chapter 3, showing by numerical simulations that the intensities of the triggered informational cascades confirm our predictions made through the tools of pinning control theory. Eventually, in Chapter 7, we will explain how the concept of phase transitions may be related to the analysis of partial pinning controllability of complex networks. Moreover, we will propose an analytic method, based on the generating functions, to predict the partial pinning controllability of a generic network, and will highlight how this approach allows to answer some related questions of the topic, while other questions are still open, and could be object of future research.

We highlight that the content of Chapter 3 has been proposed in [38], while the results of Chapters 5 and 6 are included in [57].

Selected Topics on Pinning Control of Complex Networks

Complex networks are currently being studied across many fields of science, as many systems in nature can be described by models of complex networks. Examples are numerous. The World Wide Web is a network of websites. The brain is a network of neurons. An organization is a network of people. The global economy is a network of national economies, which are themselves networks of markets, and markets are themselves networks of interacting producers and consumers.

The complex networks paradigm allows to model such real world complex systems as ensembles of dynamical systems, the *nodes*, interacting with each other according to an underlying topology [116, 124].

One of the main features of complex network paradigm is that it does not necessarily requires a purely mathematical study of the dynamical system which describes the nodes' dynamics in order to understand the network behavior. Namely, under appropriate assumptions on the nodes'

intrinsic dynamics and on the coupling mechanism, it allows to analyze the nodes' behavior just taking into account the network topology. In the view of this, in the following we summarize some graph theoretical tools necessary to cope with networked systems.

2.1 Elements of Graph Theory

A network of agents is commonly represented by a *graph*.

Definition 2.1.1. *A weighted graph \mathcal{G} is a triplet $\{\mathcal{V}, \mathcal{E}, A\}$, where $\mathcal{V} = \{1, \dots, N\}$ is the set of nodes, or vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges connecting the nodes, and $A = \{a_{ij}\}_{i,j \in \mathcal{V}} \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix, whose generic element is defined as*

$$a_{ij} \begin{cases} > 0, \text{ if } (i, j) \in \mathcal{E} \\ = 0, \text{ otherwise.} \end{cases} \quad (2-1)$$

Definition 2.1.2. *A graph is undirected if A is symmetric and its diagonal elements are equal to 0.*

Definition 2.1.3. *A graph is directed if A is not symmetric. A directed graph is also called digraph.*

In this thesis, we will mainly focus on digraphs and weighted digraphs, which include the particular cases of undirected and non weighted graphs. An undirected and non weighted graph, indeed, can be regarded as a weighted digraph with all weights equal to 1, and $A = A^\top$. In other words, the associated adjacency matrix is a binary and symmetric matrix. Notice that an unweighted graph is generally defined as the pair $\{\mathcal{V}, \mathcal{E}\}$.

Definition 2.1.4. *The set of in- and out- neighbors of a node i are defined as $\mathcal{N}_{in}(i) = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$, and $\mathcal{N}_{out}(i) = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$, respectively.*

Definition 2.1.5. *the weighted in- and out- degree of a node i are defined as*

$$d_{in}(i) = \sum_{j=1}^N a_{ji}, \quad d_{out}(i) = \sum_{j=1}^N a_{ij}, \quad (2-2)$$

respectively.

Definition 2.1.6. *The weighted in- and out- degree matrices are defined as*

$$D_{in} = \text{diag}(A^T \mathbf{1}_N) = \begin{bmatrix} d_{in}(1) & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & d_{in}(N) \end{bmatrix};$$

$$D_{out} = \text{diag}(A \mathbf{1}_N) = \begin{bmatrix} d_{out}(1) & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & d_{out}(N) \end{bmatrix}.$$

Definition 2.1.7. *A self-loop is an edge from a node to itself. Consistently with a customary convention, self-loops are not allowed in graphs.*

Definition 2.1.8. *A source is a vertex with 0 in-degree; while a sink is a vertex with 0 out-degree.*

2.1.1 Paths and Connectivity in Digraphs

Definition 2.1.9. *A directed path is an ordered sequence of vertices such that any pair of consecutive vertices in the sequence is a directed edge of the digraph. A directed path is simple if no vertex appears more than once in it, except possibly for the initial and final vertex.*

Definition 2.1.10. *A digraph is connected if there exists a path between any two vertices. If a digraph is not connected, it is composed by multiple connected components.*

Definition 2.1.11. *A digraph $\mathcal{H} = \{\mathcal{V}', \mathcal{E}'\}$ is a subgraph of $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ if $\mathcal{V}' \subseteq \mathcal{V}$, and $\mathcal{E}' \subseteq \mathcal{E}$. \mathcal{H} is a spanning subgraph if it is a subgraph and $\mathcal{V}' = \mathcal{V}$.*

Definition 2.1.12. *A cycle is a simple directed path which starts and ends in the same vertex. It is customary to accept as feasible cycles in digraphs also self-loops, and cycles of length 2.*

Definition 2.1.13. *A directed acyclic graph (DAG) is a graph which does not encompass cycles.*

Every DAG has at least one source and one sink.

Definition 2.1.14. *A directed tree is a DAG with the following property: there exist a vertex, called the root, such that any other vertex can be reached by one and only one directed path starting by the root.*

2.1.2 Connectivity Properties in Digraphs

Here, we summarize some basic properties of digraphs.

Property 2.1.1. *A digraph \mathcal{G}*

- (i) is strongly connected if there exists a directed path from any node to any other node;*
- (ii) is weakly connected if the undirected version of the graph is connected;*
- (iii) possesses a globally reachable node if one of its nodes can be reached from any other node through a directed path;*
- (iv) possesses a directed spanning tree if one of its node is the root of directed paths to every other node.*

2.1.3 Condensation Digraphs

Definition 2.1.15. *A subgraph \mathcal{H} is a strongly connected component (SCC) of \mathcal{G} if \mathcal{H} is strongly connected and any other subgraph of \mathcal{G} strictly containing \mathcal{H} is not strongly connected.*

Definition 2.1.16. A root strongly connected component (RSCC) of a graph \mathcal{G} is a SCC of \mathcal{G} such that there are no edges entering a node of the SCC that exit from a node that is not encompassed in the SCC.

Definition 2.1.17. The DAG condensation of a digraph \mathcal{G} , denoted as \mathcal{G}^D , is defined as follows: the nodes of \mathcal{G}^D are the strongly connected components of \mathcal{G} , and there exists a directed edge in \mathcal{G}^D from node \mathcal{H}_1 to node \mathcal{H}_2 if and only if there exists a directed edge in \mathcal{G} from a node of \mathcal{H}_1 to a node of \mathcal{H}_2 .

An example of condensation digraph is reported in Fig. 2-1.

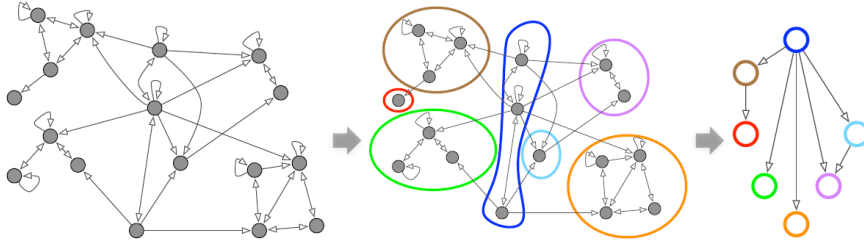


Figure 2-1: A digraph, its strongly connected components and its condensation [20].

Lemma 2.1.1. For a digraph \mathcal{G} and its condensation digraph \mathcal{G}^D ,

- (i) \mathcal{G}^D is acyclic;
- (ii) \mathcal{G} is weakly connected if and only if \mathcal{G}^D is weakly connected, and
- (iii) the following statements are equivalent:
 - a) \mathcal{G} contains a globally reachable node,
 - b) \mathcal{G}^D contains a globally reachable node, and
 - c) \mathcal{G}^D contains a unique sink.

Proof. See [20], Sec. 3.3.

2.1.3.1 Structure of a Digraph

The structure of a digraph is best summarized in the bow-tie diagram introduced in [19, 42] (see Fig. 2.1.3.1). Namely, a general digraph is composed by:

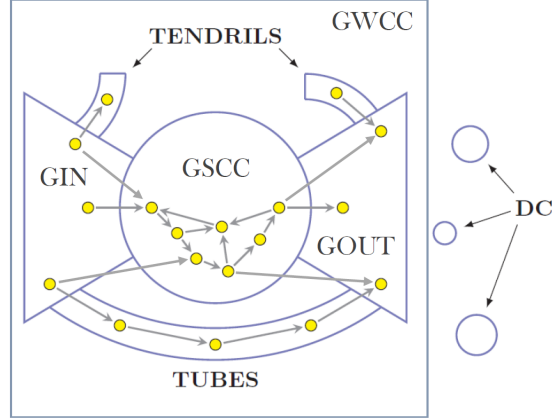


Figure 2-2: The bow-tie diagram of a digraph.

- a *Giant Strongly Connected Component (GSCC)*, in which there exists a direct path between any two nodes;
- a *Giant In-Component (GIN)*, composed by all the nodes that can reach the GSCC by a direct path, but not vice versa;
- a *Giant Out-Component (GOUT)*, each node of which is accessible starting from the GSCC;
- *tendrils*, that is, vertices which do not have direct access to the GSCC and are not reachable from it (among them, there are the *tubes*, going from the GIN to GOUT without passing through the GSCC);
- some *Disconnected Components (DC)*.

Notice that the undirected version of the bow-tie diagram consists of a *Giant Weakly Connected Component (GWCC)*, in which there exists a path between any two nodes, and disconnected components.

2.1.4 Algebraic Graph Theory

Here, we summarize some basic results which involve the correspondence between digraphs and adjacency matrices.

Property 2.1.2. (*Properties of weighted digraphs.*)

- (i) A weighted digraph \mathcal{G} is weight-balanced if and only if $A\mathbf{1}_N = A^\top \mathbf{1}_N$, i.e., $D_{in} = D_{out}$. This means that each vertex has the same weighted in- and out-degree, even if distinct vertices have distinct weighted degrees.
- (ii) In a digraph without self-loops, node i is a source if and only if $\sum_{j=1}^N a_{ji} = 0$.
- (iii) In a digraph without self-loops, node i is a sink if and only if $\sum_{j=1}^N a_{ij} = 0$.
- (iv) A is row-stochastic if and only if each node of \mathcal{G} has weighted out-degree equal to 1, that is, $D_{out} = I_N$.
- (v) A is doubly-stochastic if and only if each node of \mathcal{G} has weighted in- and out-degree equal to 1, that is, $D_{in} = D_{out} = I_N$. In this case, \mathcal{G} is weight-balanced.

2.1.4.1 Elements of Spectral Graph Theory

In this section, we provide some results on the spectral radius of a non-negative matrix A .

Theorem 2.1.1. (*Bounds on the spectral radius of a non-negative matrix.*) For a non-negative matrix A with associated digraph \mathcal{G} , the following statements hold:

- (i) $\rho(A) \leq \max(A\mathbf{1}_N)$, where $\rho(A)$ is the spectral radius of A ;
- (ii) if $\min(A\mathbf{1}_N) = \max(A\mathbf{1}_N)$, then $\rho(A) = \max(A\mathbf{1}_N)$;
- (iii) if $\min(A\mathbf{1}_N) < \max(A\mathbf{1}_N)$, the following statements are equivalent:
 - a) for each node i with $\mathbf{e}_i^\top A = \max(A\mathbf{1}_N)$, there exists a directed path in \mathcal{G} from node i to a node j with $\mathbf{e}_j^\top A < \max(A\mathbf{1}_N)$,
 - b) $\rho(A) < \max(A\mathbf{1}_N)$.

Proof. See [20].

2.1.4.2 The Laplacian Matrix

Definition 2.1.18. The Laplacian matrix of a weighted digraph \mathcal{G} is

$$L = \{l_{ij}\}_{i,j \in \mathcal{V}} = D_{out} - A, \quad (2-3)$$

where the generic element is

$$l_{ij} = \begin{cases} -a_{ij}, & \text{if } i \neq j, \\ \sum_{j=1, j \neq i}^N a_{ij} & \text{if } i = j. \end{cases} \quad (2-4)$$

In what follows, we remark some properties of the Laplacian matrix and its associated digraph.

Property 2.1.3. (*Properties of the Laplacian matrix.*)

- (i) L does not depend on the existence and weight of self-loops in \mathcal{G} ;
- (ii) \mathcal{G} is undirected if and only if L is symmetric;
- (iii) $l_{ii} = 0$ if and only if node i has zero out-degree;
- (iv) L is irreducible [14] if \mathcal{G} is strongly connected.

2.2 Synchronization of Discrete-Time Networked Systems

One of the most interesting and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes [123, 122, 6]. Namely, it has been demonstrated that many real-world problems have close relationships with network synchronization, such as the lighting of fireflies, and the spread of an epidemic or computer virus. Over the past years, the synchronization of networks had been deeply researched by many scientists from various fields, for instance, sociology, biology, mathematics and physics.

In the following, we review the main results for synchronization of discrete-time systems. Notice that, typically, consensus involves network of linear systems, while synchronization refers to networked non-linear systems.

2.2.1 Consensus of Linear Dynamical Systems

In networks of agents, consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents. A consensus algorithm (or protocol) is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network.

Let us consider a network of decision-making agents with linear dynamics

$$x_i(k+1) = x_i(k) + \epsilon u_i(k), \quad (2-5)$$

where $\epsilon > 0$, and $u_i(k)$ is the consensus protocol to be defined.

Definition 2.2.1. *We say that a consensus protocol guarantees asymptotic consensus of system (2-9) if the following holds: for every $x(0)$, there exists some c such that $\lim_{k \rightarrow \infty} x_i(k) = c, \forall i$.*

Definition 2.2.2. *We say that a consensus protocol guarantees asymptotic average-consensus of system (2-9) if the following holds: for every $x(0)$, there exists $c = (\sum_i x_i(0))/N$ such that $\lim_{k \rightarrow \infty} x_i(k) = c, \forall i$.*

For a fixed weighted topology \mathcal{G} , the following consensus protocol is used [98]:

$$u_i(k) = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)), \quad (2-6)$$

which means that each node updates its current state $x_i(k)$ to some weighted linear combination of its neighbors values. In the view of this, the dynamics of agent i becomes

$$x_i(k+1) = x_i(k) + \epsilon \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)), \quad (2-7)$$

while the collective dynamics of the network under this algorithm can be written as

$$x(k+1) = Px(k), \quad (2-8)$$

where $P = I_N - \epsilon L$ is the *Perron matrix* of the graph \mathcal{G} with parameter ϵ .

Theorem 2.2.1. *Consider the networked system in (2-7), where $0 < \epsilon < 1/\Delta$, and Δ is the maximum degree of the network \mathcal{G} , which is supposed to be strongly connected. Then*

- (i) *a consensus is asymptotically reached for all initial states;*

- (ii) the group decision value is $\alpha = \sum_i \alpha_i x_i(0)$, with $\sum_i \alpha_i = 1$;
- (iii) if \mathcal{G} is weight-balanced (or P is doubly-stochastic), an average consensus is asymptotically reached, and $\alpha = (\sum_i x_i(0))/N$.

Proof. See the proof of Theorem 2 in [97].

Thus, consensus protocol (2-6) guarantees convergence to a collective decision via local interactions for a system of linear dynamical systems coupled through a fixed topology.

2.2.2 Synchronization of Nonlinear Dynamical Systems

Let us consider the general nonlinear system

$$x_i(k+1) = f_i(x_i(k)) + \sigma \sum_j a_{ij}(k) h_{ij}(x_i(k), x_j(k)), \quad (2-9)$$

where $x_i(k)$ is the state of the i -th node of the network, $f_i(x_i(k))$ is the vector field describing its intrinsic dynamics, $h_{ij}(x_i(k), x_j(k))$ is the function that defines the interaction between nodes i and j , the coefficient $a_{ij}(k)$ indicates whether the current dynamics of node i depends on that of node j , and $\sigma > 0$ is the coupling strength. We emphasize that, in this thesis, we will mainly focus on discrete-time dynamical systems with diffusive coupling.

Definition 2.2.3. System (2-9) is said to be asymptotically synchronized if $\lim_{k \rightarrow \infty} \|x_i(k) - x_j(k)\| = 0$, or, equivalently, $x_1(k) = x_2(k) = \dots = x_N(k)$ as $k \rightarrow \infty$.

In the field of nonlinear dynamical systems, different approaches have been proposed to study the synchronization of systems as in (2-9), that is, Lyapunov–Krasovskii direct method, the Master Stability Function approach [104], contraction theory [110].

Although analytic conditions for making all nodes converge towards a synchronous asymptotic solution have been obtained, a major problem still remains from a control viewpoint. Indeed, such common solution, if it exists, cannot be determined a priori to be some desired trajectory. A possible strategy to achieve this goal would be to directly add some feedback control input on each of the systems in the network so to steer the dynamics of each individual agent towards the desired trajectory. In practice, when more than a handful of agents are considered, this approach is not viable. A feasible alternative is represented by the so-called Pinning Control Strategy [123, 78], where the control action is exerted through an additional node which is connected to a subset of the network nodes, the pinned nodes, and thus triggers the propagation of the control signals to the uncontrolled nodes through the network edges.

In the rapidly growing literature on pinning control, considerable research efforts have been focused on the analysis of coupled continuous dynamical systems [106, 52, 138, 36, 35, 32, 135]. However, pinning controllability of discrete dynamical systems is a relatively untapped research area with our ability to effectively control the evolution of such systems being currently limited to few scenarios, which will be discussed in the next section.

2.3 Pinning Control

Pinning control is a control strategy which allows to achieve synchronization of a network of dynamical systems towards a desired trajectory. This strategy assumes the existence of an external node, *the pinner*, which generates a reference trajectory used to exert a control action only to a small fraction of the networked nodes, also called *pinned nodes*. In this scenario, the problem consists not only in designing the strength

and form of the control action to be exerted by the pinner, but also in determining how many, and what type of pinned nodes need to be selected to achieve the control objective [113].

Let us consider the nonlinear system

$$x_i(k+1) = f(x_i(k)) + \sigma \sum_{j=1}^N l_{ij}(k) h(x_j(k)), \quad (2-10)$$

where f and h are nonlinear functions, $l_{ij}(k)$ is the generic element of the time-varying laplacian matrix, and the pinner's trajectory \bar{x} satisfying

$$\bar{x} = f(\bar{x}). \quad (2-11)$$

Definition 2.3.1. *Network (2-10) is said to be fully pinning controlled to the pinner's trajectory (2-11) when $\lim_{k \rightarrow \infty} \|x_i(k) - \bar{x}\| = 0 \forall i$.*

To achieve synchronization, feedback pinning controllers are applied to a subset \mathcal{P} of the network nodes, where $|\mathcal{P}| = p$. In the view of this, the controlled network can be described as

$$x_i(k+1) = f(x_i(k)) + \sigma \sum_{j=1}^N l_{ij}(k) h(x_j(k)) + u_i(k), \quad (2-12)$$

with the local feedback controllers given by

$$u_i(k) = -\kappa \delta_i (h(x_i(k)) - h(\bar{x})), \quad (2-13)$$

where κ is the control gain, and

$$\delta_i = \begin{cases} \delta, & \forall i \in \mathcal{P}, \\ 0, & \text{otherwise.} \end{cases} \quad (2-14)$$

Then, network (2-10) becomes

$$x_i(k+1) = f(x_i(k)) + \sigma \sum_{j=1}^N l_{ij}(k) h(x_j(k)) - \kappa \delta_i (h(x_i(k)) - h(\bar{x})). \quad (2-15)$$

Unfortunately, while the literature on pinning control of continuous-time systems is extremely vast, see e.g. [65, 79, 114, 134] and the references therein, only few contributions are available in the literature which study the pinning controllability of system (2-15). Among these, in [132], the authors generalize the classical master stability function introduced in [103] to provide a necessary and sufficient condition for local pinning controllability of discrete-time systems. The results are extended to the case of uniform constant communication delays in [82]. In the recent work [131], instead, sufficient conditions based on Lyapunov function for achieving synchronization of discrete-time networks via impulsive pinning control are provided, while [94] investigates the controllability of discrete-time networks of coupled chaotic maps through stochastic pinning. Eventually, [136] shows through extensive simulations that pinning control of discrete-time dynamical networks is a challenging task, whereby direct control of a large fraction of the network nodes is generally required. Notice that all the aforementioned studies account for the hypothesis of static topologies. To the best of our knowledge, no contributions have been made in the field of pinning control of discrete-time systems with time-varying networks of interaction.

2.4 Partial Pinning Control

Often, in applications, achieving complete controllability is a chimera as both economical and physical constraints typically affect the selection of the pinned nodes. For instance, previous works [81, 93] has pointed out that for gene regulatory networks, a considerable amount of pinned nodes are needed to achieve complete controllability, which can turn out unfeasible. Moreover, it is often the case that the selection of the pinned nodes is restricted to a well-defined subset of the nodes of the

network. For instance, in designing curative interventions, only some easily accessible proteins are designated as targets for drugs [71].

In this perspective, the problem becomes that of selecting the nodes to be pinned so as to drag the greatest number of nodes under control of the pinner under physical or economic constraints. This is what we call the partial pinning control problem.

Definition 2.4.1. *Network (2-10) is said to be q -partially pinning controlled to the pinner's trajectory (2-11) when $\lim_{k \rightarrow \infty} \|x_i(k) - \bar{x}\| = 0 \ \forall i \in \mathcal{Q}$, where $\mathcal{Q} \subseteq \mathcal{V}$, and $q = |\mathcal{Q}|$.*

In [37], the partial pinning control problem is defined as

$$\begin{aligned} q^* &= \max_{\mathcal{P}} |\mathcal{Q}| \\ |\mathcal{P}| &= p. \end{aligned} \tag{2-16}$$

The problem is solved for a class of continuous nonlinear systems, and for any limited numbers of pinned nodes. Namely, under appropriate assumptions on the nodes' dynamics, structural conditions which ensure the partial pinning controllability of the system are provided. The problem is translated into an integer linear program (ILP), and an optimization problem is formulated and solved for the selection of the pinning and coupling gains.

Unfortunately, no analytic results in the literature accounts for the problem of partial pinning controllability of discrete dynamical systems.

That's why, in this thesis, we approach to partial pinning control focusing on the structural conditions available in [37], assuming that the conditions on the nodes' dynamics are satisfied.

2.4.1 Structural Conditions for Partial Pinning Controllability

In order to find the topological conditions to maximize the number of nodes which synchronize to the pinner's trajectory, the pinned node selection algorithm is proposed in [37]. It translates problem (2-16) of maximizing the number of pinning controlled nodes q into an optimization problem on a graph. To do so, it relies on the following structural condition: a node is pinning controllable if all of the RSCC in its upstream encompass at least a pinned node. This condition is based on the following property of the RSCC:

Property 2.4.1. *A RSCC is pinning controllable if at least one node belonging to the RSCC is pinning controllable.*

Hence, the algorithm selects the set of p RSCCs to be pinned that maximize the number of nodes fulfilling such structural condition.

In what follows, we remark the main steps of the partial pinning control algorithm.

Let us denote:

- \mathcal{G}^D the DAG condensation of a generic digraph \mathcal{G} ;
- \mathcal{C} the set of pinnable nodes, with $\mathcal{P} \subseteq \mathcal{C}$;
- γ_i the generic SCC of \mathcal{G}^D ;
- r_i the generic RSCC of \mathcal{G}^D ;
- $\Gamma(r_i)$ the set of nodes in the downstream of r_i ;
- Φ the set of pinned RSCCs.

In the view of this, problem (2-16) becomes

$$\Phi^* = \max_{|\Phi|=p} |\Gamma(\Phi)|. \quad (2-17)$$

Case $p = 1$

By pinning an arbitrary node of r_i , all the nodes in $\Gamma(r_i)$ are pinning controlled. Hence, if $p = 1$, the solution of problem (2-17) consists in selecting as the only pinned node an arbitrary node of RSCC r_{i^*} , with $i^* = \arg \max_i |\Gamma(r_i)|$.

Case $p > 1$

The procedure is the following:

- 1) build a new graph \mathcal{G}' through the following steps:
 - (a) define \mathcal{R}_C as the set of roots of \mathcal{G}^D that include at least a node belonging to \mathcal{C} ;
 - (b) add to \mathcal{G}' all the roots in \mathcal{R}_C ;
 - (c) add to \mathcal{G}' all the non-roots γ_i of \mathcal{G}^D that are in the downstream of no more than p roots of \mathcal{G}^D (all encompassed in \mathcal{R}_C);
 - (d) for all pairs $\gamma_i, r_j \in \mathcal{G}'$, add an edge y_{ij} connecting γ_i to r_j if, in \mathcal{G}^D , γ_i is in the downstream of r_j ;
 - (e) add an additional node π , representing the pinner, and connect it to all the roots $r_j \in \mathcal{G}'$ through a set of edges $y_{j\pi}$;
- 2) associate to all edges of the graph \mathcal{G}' the following weights:
 - (a) $w_{ij} = |\gamma_i| \ \forall i$, i.e., all edges entering the i -th node γ_i have a weight equal to the number of nodes in the SCC γ_i ;

(b) $w_{j\pi} = |r_j| \forall j$, i.e., all edges entering the j -th root r_j have a weight equal to the number of nodes in the RSCC r_j ;

3) solve the following ILP:

$$\max_y \sum_i \sum_j w_{ij} y_{ij} + \sum_j w_{j\pi} y_{j\pi} \quad (2-18)$$

s.t.

$$\sum_j y_{j\pi} = p \quad (2-19)$$

$$\sum_j y_{ij} \leq \frac{1}{\deg_{in}(i)} \sum_{j|\exists y_{ij}} y_{j\pi} \quad \forall i \quad (2-20)$$

$$y_{ij}, y_{j\pi} \in \{0, 1\} \quad \forall i, j \quad (2-21)$$

This procedure first creates a new graph \mathcal{G}' , whose nodes are RSCCs encompassing at least a node of \mathcal{C} or SCCs in the downstream of such RSCCs. Each node representing a RSCC is connected to the SCCs in its downstream. Notice that the procedure does not include any node representing an SCC having either (i) more than p RSCCs in its upstream or (ii) an RSCC in its upstream that does not encompass any node of \mathcal{C} , as it cannot guarantee that these SCCs are pinning controlled with a selection of p pinned nodes. Then, the pinner π is added to \mathcal{G}' and is connected to all nodes r_i representing the RSCCs. Finally, it associates to each edge in \mathcal{G}' a weight equal to the number of nodes in the (R)SCC it points to. The solution of the ILP in (2-18)-(2-21) is then equivalent to determine the RSCCs in which a pinned node must fall and the corresponding SCCs that can be pinning controlled. Namely, SCC γ_i can be pinning controlled if there exists a j such that $y_{ij} = 1$, and RSCC r_j will include a pinned node if $y_{j\pi} = 1$. Accordingly, the objective function to be maximized in (2-18) represents the total number of pinning controlled nodes. The constraint (2-19) guarantees that the

pinned nodes are p , while Eq. (2-20) imposes that the nodes of an SCC are pinning controlled only if a node is pinned in each of the RSCCs in their upstream.

The Artificial Financial Market

In this chapter, we build an artificial financial market, which will be used for testing different models of agents' behavior in order to analyze and predict the possible emergence of informational cascades.

Our interest focus on agent-based modeling (ABM), as it succeeds in complementing neoclassical economics models, removing some of their main assumptions, such as homogeneity of the financial agents [2, 30]. As evidence of their effectiveness, in the last decades several contributions in the field of financial ABM have been proposed, see for instance [4, 21, 102, 26, 50, 59, 74, 75, 87, 120, 128, 111, 129, 40, 24, 99, 61, 23].

Agent-based approaches have also been used to test the effects of policies, regulations and taxation systems on the market dynamics, see for instance [129, 40]. Inspired by the seminal work of Tobin [119], for instance, several taxes on financial transactions were proposed to regulate the markets, whose effects have been controversial [62, 77].

To the best of our knowledge, none of the existing models of artificial markets accounts for herding phenomena and different taxation schemes

at the same time. In the view of this, we build a novel artificial financial market that is capable of testing the delicate interplay between agents' interactions, the inequality in wealth distribution, and the balancing of two common alternative taxation schemes.

The concepts of learning and adaptation are generally applied to utility functions, that may not be fixed, as observed in [29, 31, 33]. In the view of this, in our market the agents behave according to utility theory, and, at a first instance, are grouped in three classes with different risk attitudes and subsequent trading strategies.

We consider two kinds of agents' interaction. In the first case, the agents do not interact with each other. They are stubborn agents [105], who keep their own risk attitude regardless of the effectiveness of the consequent trading strategy. In the second case, we consider an interaction dynamics which accounts for the presence of leaders.

In order to test the likelihood of our artificial market with empirical evidence, we study the emerging features of the market in terms of trading volumes and wealth distribution, characterized through the Gini coefficient [60], in presence of a Tobin-like tax and a flat tax, respectively. Even though the validation of agent-based models is a hard task [74], we show that our results are qualitative consistent with some empirical evidences. Thus, this scenario is useful to test the emergence of herding behavior, which we will manage from a new perspective, that of control theory. The details of our choice will be extensively discussed in the following chapters.

3.1 Market Structure

We introduce an agent-based financial market populated by a set of N agents, who can choose among alternative financial assets. The state of

each agent i is defined as his current wealth r_i and risk attitude y_i . The agents behave according to the Von Neumann and Morgenstern utility theory [121], and alternative taxation schemes and interaction rules are considered.

3.1.1 Financial Assets

At each time step $k = 1, 2, \dots$, a simulated trading session is performed. Each agent, in a sequential random order, evaluates the convenience of investing a given fraction ϵ of his current wealth $r_i(k)$ in one of the financial assets from the set $\mathcal{L} = \{1, \dots, L\}$. The return of the j -th asset is modeled as a stochastic process β^j with equiprobable realizations \bar{a}^j and \bar{b}^j . In other words, \bar{a}^j and \bar{b}^j are the win and loss rates associated to the j -th asset¹. The assets in \mathcal{L} are characterized by a limited availability $A_j, j = 1, \dots, L$, where $A_L = +\infty$ is associated to a virtual asset, corresponding to no-investment. In view of this, each agent is allowed to invest in one of the available assets, that is, in any element of \mathcal{L} such that $A_j \geq \epsilon r_i(k)$. Agents' access to trading is randomly permuted at each time step k , so that, on average, no agent is favored. After each trading, the availability of the selected asset is updated before the next agent is allowed to trade.

3.1.2 Trading Strategy

According to the Von Neumann and Morgenstern utility theory [121], a rational agent who acts in an uncertain environment takes his choices maximizing the expected value of some function defined as utility function. In the view of this, we associate to each agent the following

¹Notice that $\bar{x}^j = E[\beta^j] = 0.5(\bar{a}^j + \bar{b}^j)$ is the expected return of asset j , which is assumed to be constant for the sake of simplicity.

power-law utility function:

$$E[U_i(k)] = 0.5 \left[(a_i(k)\epsilon r_i(k))^{y_i(k)} + (b_i(k)\epsilon r_i(k))^{y_i(k)} \right], \quad (3-1)$$

where $E[U_i(k)] = [E[U_i^1(k)], \dots, E[U_i^L(k)]]$ is a vector which includes the current expected utilities that agent i associates to each asset $j = 1, \dots, L$, $y_i(k)$ is the state variable which determines the risk attitude of the i -th agent, $a_i(k) = [a_i^1(k), \dots, a_i^L(k)]$ and $b_i(k) = [b_i^1(k), \dots, b_i^L(k)]$ are the current win and loss rates associated by agent i to all the assets. Equation (3-1) is a flexible function that allows us to model heterogeneous agents which differ each other in some crucial features, such as risk attitude and information on the assets.

For the sake of simplicity, at a first instance we assume that all the agents share the correct information on the assets. This means that all the agents know the correct probability distribution of the assets' return. Thus, we select $a_i(k) = \bar{a} = [\bar{a}^1, \dots, \bar{a}^L]$, and $b_i(k) = \bar{b} = [\bar{b}^1, \dots, \bar{b}^L] \forall i, k$ (in Chapter 6, this constraint will be relaxed).

In the view of this, Eq. (3-1) becomes

$$E[U_i(k)] = 0.5 \left[(\bar{a}\epsilon r_i(k))^{y_i(k)} + (\bar{b}\epsilon r_i(k))^{y_i(k)} \right]. \quad (3-2)$$

At each time instant k , agent i evaluates the possibility of investing the fraction ϵ of his current wealth based on his expected utilities of the assets. Hence, the trading decision

$$s_i(k) = g(E[U_i(k)]), \quad (3-3)$$

is a function g which returns a vector $s_i(k) = [s_i^1(k), \dots, s_i^L(k)]$ defining the ranking of the assets corresponding to agent's i preferences:

$$s_i^j(k) := m : \exists m-1 \ E[U_i^l(k)] > E[U_i^j(k)], \exists L-m \ E[U_i^l(k)] < E[U_i^j(k)]. \quad (3-4)$$

The j -th element $s_i^j(k)$ of $s_i(k)$ is the integer m defining the position of asset j in the ranking of the assets made by agent i . Due to the limited availability of the assets, the actual trading action made by agent i corresponds to trade in the first available asset l^* according to his preferences $s_i(k)$.

The outcome of the trade made by agent i is a realization of β^{l^*} . Therefore, the dynamic equation describing the evolution of the wealth $r_i^-(k)$ of agent i before the taxation is given by:

$$r_i^-(k) = r_i(k-1) + \gamma \epsilon r_i(k-1)(\bar{a}^{l^*} - 1) - (1-\gamma) \epsilon r_j(k-1)(1 - \bar{b}^{l^*}), \quad (3-5)$$

where γ is a binary variable ($\gamma = 1$ if the trading made by agent i is successful, and 0 otherwise). When the trading session is over, a tax is applied and the wealth of agent i at time k is updated as

$$r_i(k) = \tau(r_i^-(k)), \quad (3-6)$$

where τ is a nonlinear function describing the selected taxation scheme. In what follows, we characterize this function for two different taxation schemes.

3.1.3 Taxation Schemes

We consider two alternative taxation systems, which affect the current wealth of the agents $r_i^-(k)$, $i = 1, \dots, N$ in different ways: a) taxation on financial transactions, and b) taxation on wealth.

3.1.3.1 Tobin-like Tax

Taxes on financial transactions have been adopted in several countries in the last century: a well-known example is the so-called Tobin Tax [119], named after the Nobel prize James Tobin, whose original scope

was to put a penalty on short-term financial round-trip excursions into another currency.

The financial transaction tax a) is actually a Tobin-like tax (TT), which reduces the current wealth of the winning agents by a profit fraction $u(k)$ given by

$$u(k) = \begin{cases} \frac{p(k)}{\sum_{i=1}^N h_i(k)}, & p(k) > 0, \\ 0, & p(k) \leq 0, \end{cases} \quad (3-7)$$

where $h_i(k) = r_i^-(k) - r_i(k-1)$, and $p(k) = \sum_{i=1}^N (r_i^-(k) - r_i(0))$. Accordingly, (3-6) becomes

$$r_i(k) = r_i^-(k) - H(h_i(k))h_i(k)u(k), \quad (3-8)$$

where H is the Heaviside step function.

3.1.3.2 Flat Tax

The alternative taxation scheme b) is a flat tax (FT), in which the amount of the tax is proportional to the total wealth of the individual. Accordingly, (3-6) becomes

$$r_i(k) = v(k)r_i^-(k), \quad (3-9)$$

where $v(k) = \frac{\sum_{j=1}^N r_j(0)}{\sum_{i=1}^N r_i^-(k)}$.

Notice that, to allow for a proper comparison between the two taxation schemes, the coefficients $u(k)$ and $v(k)$ in (3-7) and (3-9), respectively, are selected so as to keep the average wealth constant over time, that is, $\frac{1}{N} \sum_{i=1}^N r_i(k) = \bar{r}$.

3.2 Agents' Behavior

We populate the artificial market with heterogeneous agents. Based on their current risk attitude, we group the agents in three classes. In

the first one, there are the agents characterized by a low risk attitude, denoted in what follows as *prudent* agents. The agents that are more prone to take risks are denoted *audacious* and grouped in the third class. Finally, the intermediate class groups the *ordinary* agents. We emphasize here that an agent may decide not to invest (formally, to invest in the L -th asset), if $E[U_i^L(k)] \geq E[U_i^j(k)]$ for all $j = 1, \dots, L-1$. We test our scenario by considering two different kinds of agents' behavior:

Case I

the agents do not interact with each other. Thus, the market is composed of stubborn agents, who do not modify their utility function even if they observe that their investing strategy is not successful. Accordingly, their risk attitude is considered as a parameter rather than an evolving state, and coincides with the initial risk attitude $y_i(0)$ for all $k \in \mathbb{N}$, $i = 1, \dots, N$.

Case II

The agents are adaptive, as they are prone to directly interact with each other and update their trading strategy. In particular, we model the strategy modification as a variation of the risk attitude $y_i(k)$ in (3-2). The reciprocal influence among the agents diffuses through a connection topology described by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where \mathcal{V} is the set of nodes, corresponding to the agents, and \mathcal{E} is the set of directed edges connecting the nodes (see Sec. 2.1). The existence of an edge (i, j) implies that the risk attitude of node j is influenced by that of node i . The dynamics of $y_i(k)$ in (3-2) is described by

$$y_i(k) = (1 - \lambda)y_i(0) + \frac{\lambda}{|\mathcal{N}_i|} \sum_{h \in \mathcal{N}_i} y_h(k-1), \quad (3-10)$$

where λ is the interaction weight, $y_i(0)$ is the inner risk attitude, and \mathcal{N}_i is the set of in-neighbors of agent i (see Def. 2.1.4)². We remark that the bigger the coefficient λ is, the more the agents are prone to modify their utility function: $\lambda = 0$ models the case of stubborn agents, while $\lambda = 1$ the case in which the agents completely disregard their innate risk attitudes and emulate the neighbor behaviors.

3.2.1 Leaders and Communities

The interaction topology is modelled as a disjoint directed scale-free network, and the graph \mathcal{G} is decomposed in up to three disconnected components, the *communities*, each of which is guided by *leaders* belonging to the same risk attitude class. Namely, inside each community, we consider *emulating the rich* dynamics, where the richest agents are stubborn, but they influence the other agents, so playing the role of leaders [44]. We choose to consider separated communities so that each follower cannot be influenced by leaders with significantly different risk attitudes. Accordingly, each follower elects to emulate the strategy he considers most profitable. The size of the communities is proportional to the total wealth of their leaders and, inside each community, the richest agents are more likely to activate links.

The interaction is triggered at a given time instant k_t . Henceforth, the dynamics of $y_i(k)$, $i = 1, \dots, N$, described in (3-10), are strongly influenced by the structure of the graph \mathcal{G} describing the diffusion flow. In turn, the structure of \mathcal{G} is established at time k_t , based on the current wealth $r_i(k_t)$, for $i = 1, \dots, N$.

²More details on the interpretation of the risk attitude dynamics will be further explained in 5.1.2

3.3 Emerging Features

The proposed artificial financial market can take into account different scenarios, in terms of both taxation schemes and interaction rules. In order to test our market, we focus our attention on the effects of agents' interaction on the emerging features of the market. We aim at identifying the possible interplay between taxation and interaction in determining the trading volumes and the wealth distribution among the agents, in order to check if the results are consistent with empirical evidence. To do so, we compare the results of extensive simulations of Case II against that of Case I. The effect of the alternative taxation schemes are firstly pointed out in the case with no interactions. Then, we focus on adaptive agents and study the effects induced by the emulating dynamics on both the wealth distribution and the trading volumes for both taxation schemes.

To highlight the overall wealth dispersion induced by the two taxation schemes, we use the Gini coefficient $G(k)$, proposed by Corrado Gini in [60] as a measure of inequality of income or wealth, which can be defined as

$$G(k) = 1 - \frac{2}{N-1} \left(N - \frac{\sum_{i=1}^N i r_i(k)}{\sum_{i=1}^N r_i(k)} \right), \quad (3-11)$$

where the wealths $r_i(k)$, $i = 1, \dots, N$, are indexed in non-decreasing order, that is, $r_i(k) \leq r_{i+1}(k)$. The Gini coefficient varies between 0, which reflects complete equality, and 1, which indicates complete inequality (one person holds the all wealth, all others have none).

3.4 Simulation Results

To achieve statistical relevance, we run 1000 simulations for each scenario and consider a number of time steps sufficient to reach steady-state

wealth distribution.

We consider $N = 1000$ agents that share the same initial capital $r_i(0) = 100\$$, $i = 1, \dots, N$, and, at each trading session k , can decide to invest a fraction $\epsilon = 0.2$ of their current wealth. The cardinality of the set of assets is $L = |\mathcal{L}| = 4$, that is, the agents can trade in three categories of actual assets, while the fourth one corresponds to no-investment and, therefore, has an unlimited availability. On the other hand, at every time instant, each of the three actual assets has an availability equal to 1/15th of the total wealth of the system. The win and loss rates associated to the actual assets are selected so that the prudent agents ($0.5 \leq y_i(k) < 0.67$) only consider investing in the first and less risky asset, the ordinary agents ($0.67 \leq y_i(k) < 0.83$) also consider the second, while the audacious agents ($0.83 \leq y_i(k) \leq 1$) also find convenient investing in the third and most risky one. Namely, the won rates are $\bar{a} = [1.53, 1.60, 1.67, 1]$, while the loss rates are $\bar{b} = [0.60, 0.50, 0.40, 1]$. The initial risk attitudes $y_i(0)$, $i = 1, \dots, N$, are uniformly distributed in the interval $[0.5, 1]$.

Case I

In our numerical analysis, we monitor the effects of the alternative taxation schemes on both the wealth distribution and the trading volumes in the artificial market. Specifically, we observe that the TT scheme hinders the audacious agents, favoring the prudent ones. This is clearly depicted in Figure 3-1(a), which shows the sum of the average final wealth in the three classes of agents, respectively. The opposite is observed when a flat tax is adopted, in which the wealth distribution is biased towards the ordinary and audacious agents, see Figure 3-1(b). In other words, the TT scheme does not reward the risk, penalizing the audacious agents, in opposition to the FT scheme, which encourages the

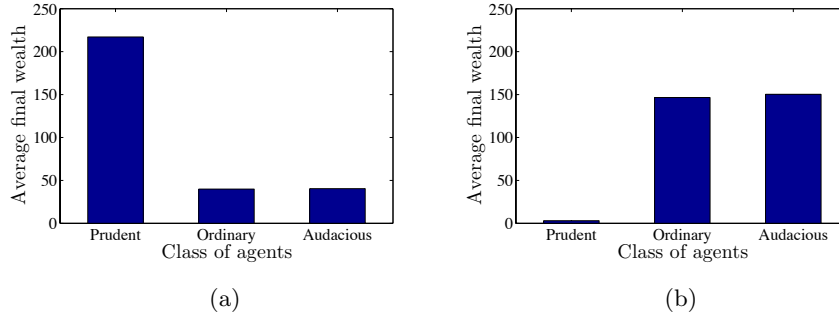


Figure 3-1: Case I. Final wealth distribution when TT (a) and FT (b) schemes are introduced, respectively.

agents to trade. As depicted in Figure 3-2(a), while the TT scheme induces a wealth redistribution among the population, the FT scheme increases inequalities. On the other hand, the TT scheme leads to lower trading volumes at the steady-state, see Figure 3-2(b). The latter effect is in line with the criticisms commonly made to financial transaction taxes, which are blamed for possible market depression [10, 62, 86].

Case II

We assume that, after k_t time steps in which the agents invest based on their own risk attitude, the emulation dynamics described in (3-10) are triggered. The triggering instant of the emulation behaviour is indifferent to our purpose, as alternative values of k_t only affect the size of the communities, see Figure 3-3. The ten richest agents (the leaders) are assumed to have only outgoing edges; the followers, instead, have bidirectional edges with their neighbouring peers, and may have ingoing edges from the leaders. To model this interaction mechanism, we build a directed Barabási-Albert (BA) scale-free network [12], in which the hubs coincide with the leaders. The network is split into three communities,

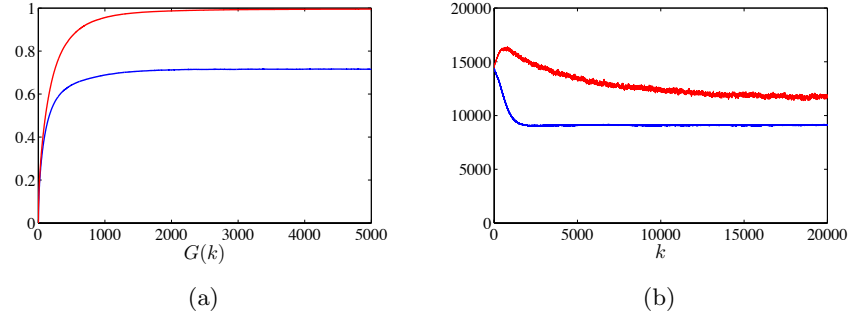


Figure 3-2: Case I. Gini coefficient (a) and trading volumes (b) when TT (blue line) and FT (red line) schemes are introduced, respectively.

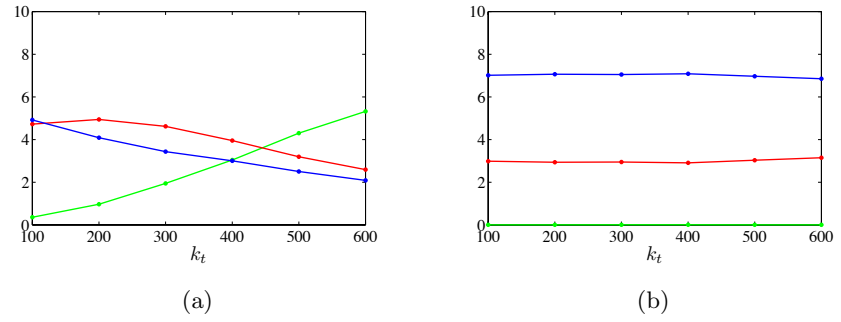


Figure 3-3: Case II. Average number of leaders of belonging to the communities 1 (red line), 2 (blue line) and 3 (green line) for different interaction triggering time k_t when TT (a) and FT (b) schemes are introduced, respectively.

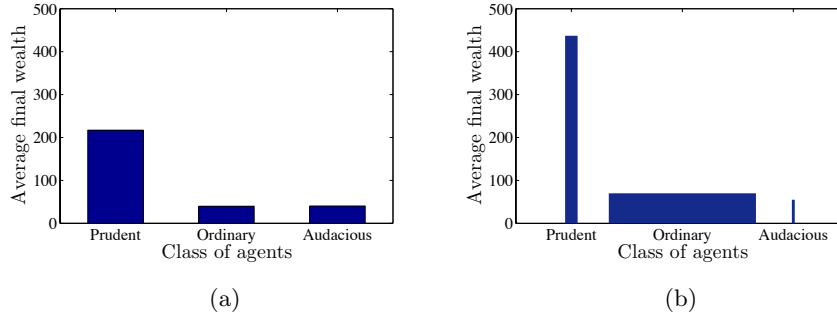


Figure 3-4: Final wealth distribution when TT scheme is introduced: Case I (a) and Case II (b), respectively. The width of the bars is proportional to the numerosity of the classes.

led by the prudent, ordinary, and audacious leaders, respectively. The size of each community is proportional to the sum of the leaders' wealth, which is an indirect measure of their influence.

The results are compared against a twin set of simulations in Case I, sharing the same set of realizations of the stochastic processes $\beta^j, j \in \mathcal{L}$, for each trader, and at each time instant. For the sake of clarity, we analyze the two taxation schemes separately, starting with the TT scheme. From Figure 3-4, we observe that this taxation system, regardless of the interactions among the agents, recompenses the prudent strategies in the long run. The emulating the rich interaction, instead, skews the distribution among the communities, see Figure 3-5.

In absence of leaders, the interaction with the neighbors would tend to average the agents' attitudes towards risk. However, the presence of leaders differentiates the communities. In particular, the community guided by the prudent leaders preserves a significant number of prudent agents (the 14% of the total population of the community, see Table 3.4). Consequently, the average wealth of the agents in the first community

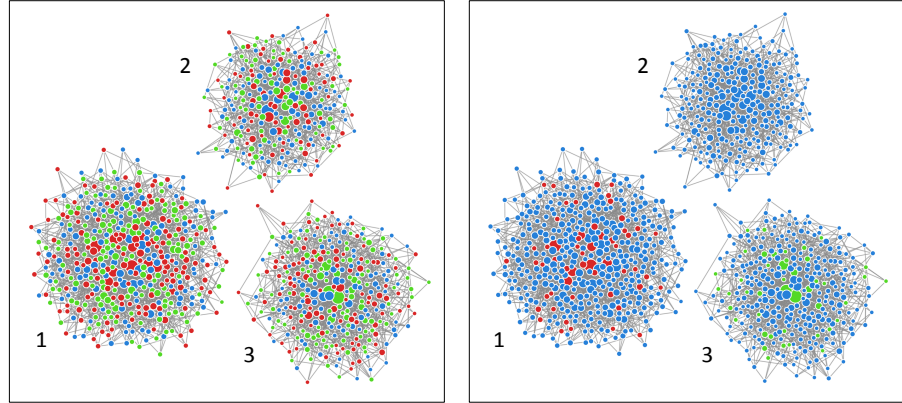


Figure 3-5: Case II, TT scheme. Screenshots of the three communities before (left panel) and after (right panel) the agents' interaction in a sample simulation. The red, blue, and green nodes correspond to prudent, ordinary, and audacious agents, respectively. Notice that the averaging of the attitudes increases the overall density of ordinary agents. However, the leaders' influence skews the distribution across the communities, with prudent and audacious agents still populating the first and third community, respectively.

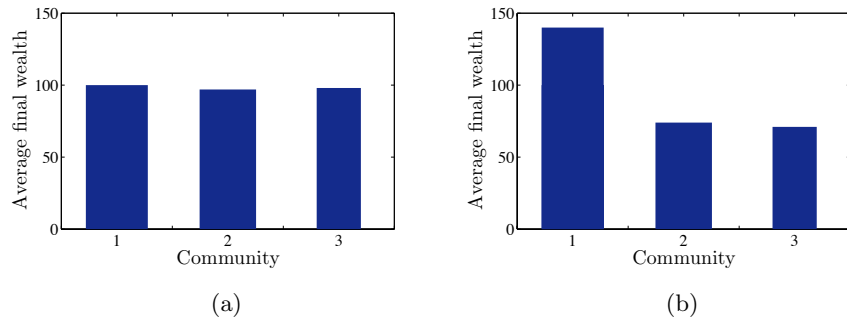


Figure 3-6: Average final wealth of the agents belonging to communities 1, 2 and 3, when TT scheme is introduced: Case I (a) and Case II (b), respectively.

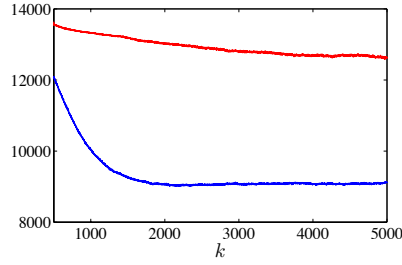


Figure 3-7: Trading volumes when TT scheme is introduced: Case I (blue line) and Case II (red line), respectively.

is considerably higher compared with the other two communities (see Figure 3-6(b)), whose leaders pursue risky strategies which turn to be unprofitable in the long term (their steady-state capital is lower than the average agents' wealth $\bar{r} = 100$, see Table 3.4).

Moreover, Figure 3-7 shows that the emulation mechanism has the further effect of mitigating the decrease in trading volumes typical of the TT case. This is due to the reduction of the total number of prudent agents illustrated in Figure 3-4.

Differently from the TT Case, in which the prudent agents slowly but relentlessly take the leadership of the market, see Figure 3-3(a), when the flat tax is introduced, a prudent strategy is disadvantageous both in the short and in the long term, see Figure 3-3(b). Consequently, no agent is encouraged to emulate the prudent agents, and the market splits into only two communities, guided by the ordinary and audacious leaders, respectively. Accordingly, there are no notable differences in the average wealth of the two communities, see Table 3.4. We emphasize that, while the emulating dynamics can strongly influence the distribution of the wealth across the communities, the overall wealth distribution is only dictated by the taxation scheme. In particular, the variation of the Gini coefficient induced by the emulation dynamics is an order of

magnitude lower than that induced by a change of taxation scheme, for any possible value of the interaction weight λ and of the average degree of the connection topology.

We remark that considering disconnected communities is an idealization of real-world aggregations, where few weak links may still connect the communities. However, all the presented results are robust to the addition of links connecting the communities. This is confirmed by a twin set of simulations in which a small fraction (less than 5%) of the network edges are rewired following a degree-preserving procedure inspired by the work in [73]. Considering 95% confidence intervals, we find that the variations of the results are not statistically significant.

Summing up, the numerical analysis replicates the well known benefits and drawbacks of the two taxation schemes, and the analyzed emerging features are in line with empirical evidence. Namely, we observe a trade-off between wealth redistribution and trading volumes: while the Tobin-like tax has the effect of redistributing the wealth among the agents, but reduces the trading volumes, the opposite happens with a flat tax, which encourages to invest, but dramatically increases the disparity among the agents. Moreover, while the TT scheme favors the prudent agents investing only in the less risky assets, the FT scheme rewards the audacious agents, that also consider investing in the riskiest assets. In the other case, where the adaptive agents consider adjusting their risk attitude and the consequent trading strategy, we observe a significant impact of the agents interactions on the emerging features of the market. Indeed, the richest agents, recognized as the market leaders, form separate communities. Notably, the communities benefit from the presence of leaders with successful trading strategies, and are more likely to increase their average wealth. Moreover, this imitation behavior mitigates the reduction of the trading volumes typical of Tobin-like

	Tobin Tax		
Community	1	2	3
$\bar{r}_I(T)/\bar{r}$	1.01 [0.99, 1.02]	0.97 [0.96, 0.99]	0.98 [0.97, 1.00]
$\bar{r}_{II}(T)/\bar{r}$	1.40 [1.38, 1.42]	0.74 [0.73, 0.75]	0.71 [0.70, 0.73]
ν_{c_i}	380.76 [369.89, 391.01]	347.63 [336.54, 357.92]	271.61 [260.91, 281.71]
$f_1(\%)$	20.29 [19.78, 20.80]	0.60 [0.53, 0.66]	0.01 [0.01, 0.02]
$f_2(\%)$	79.71 [79.20, 80.21]	99.40 [99.32, 99.45]	93.04 [92.67, 93.41]
$f_3(\%)$	0	0	6.95 [6.58, 7.32]

	Flat Tax		
Community	1	2	3
$\bar{r}_I(T)/\bar{r}$	0	1.07 [1.01, 1.13]	0.97 [0.94, 1.01]
$\bar{r}_{II}(T)/\bar{r}$	0	1.05 [0.99, 1.11]	0.98 [0.95, 1.01]
ν_{c_i}	0	246.24 [235.76, 256.19]	753.76 [742.78, 763.20]
$f_1(\%)$	0	0.80 [0.69, 0.92]	0
$f_2(\%)$	0	99.16 [99.04, 99.27]	93.13 [92.95, 93.32]
$f_3(\%)$	0	0.04 [0.03, 0.06]	6.87 [6.68, 7.05]

Table 3-1: $\bar{r}_I(T)$ and $\bar{r}_{II}(T)$ are the average final wealth in Case I and Case II, respectively; ν_{c_i} is the average numerosity of the i -th community, $i = 1, 2, 3$; f_j is the final percentage of agents belonging to the j -th class, $j = 1, 2, 3$. Confidence intervals with significance level 0.05 are also reported.

	Leaders	L1	L2	L3
Tobin Tax	$\bar{r}_l(T)/\bar{r}$	7.69 [7.18,8.19]	0.66 [0.59,0.72]	0.64 [0.37,0.90]
	ν_{l_i}	4.30 [4.20,4.40]	3.20 [3.10,3.29]	2.50 [2.41,2.59]
Flat Tax	$\bar{r}_l(T)/\bar{r}$	0	14.29 [12.60,15.98]	14.48 [13.24,15.71]
	ν_{l_i}	0	3.03 [2.94,3.12]	6.97 [6.88, 7.06]

Table 3-2: $\bar{r}_l(T)$ is the average final wealth of the leaders in the focal scenario; ν_{l_i} is the average numerosity of the leaders of the i -th community, $i = 1, 2, 3$. Confidence intervals with significance level 0.05 are also reported.

taxes, while preserving its redistributive effect.

Informational Cascades: an Overview

We are influenced by others in almost every activity, and this includes investments and financial transactions. Such influence may be entirely rational, but investors are often accused of irrationally converging their actions and beliefs.

The word “herding” is defined to include any behavior similarity brought by the interaction of individuals. People’s thoughts, feelings, and actions can be influenced by others by several means, as observations of actions, observations of the consequences of actions, such as individual’s payoffs, direct communications, reputations, and so on.

Payoff externalities, indeed, may drive the decisions of agents for which stocks they acquire information. Under certain circumstances, agents find it worthwhile to acquire further information only if other agents do. Agents thus herd on information acquisition (or lack thereof). Herding can also be caused by principal-agent concerns. Managerial performance evaluation is often based on relative not absolute performance. Typically, this behavior show that each agent prefers to mimic the actions of

other traders, completely ignoring private information, to avoid being revealed to be of low-ability [39]. This process of imitation and social learning may lead the agents in a blind replication of the same actions. This phenomenon is generally known as “informational cascade”. They describes a condition in which imitation will occur with certainty. An individual is said to be in an informational cascade if, based upon his observation of others (e.g., their actions, outcomes, or words), his selected action does not depend on his private information signal [15]. In such a situation, his action choice is uninformative to later observers. Thus, cascades tend to be associated with information blockages. Even a simple form of social interaction as imitation offers a crucial benefit: it allows an individual to exploit information possessed by others about the environment. Indeed, the idea is that people gain useful information from observing previous agents’ decisions, to the point where they optimally and rationally completely ignore their own private information. In the literature, the notions of informational cascades and herd behavior are often considered equivalent, but this two concepts are quite distinct. Banerjee, for instance, uses the term “herd” for what we refer as a cascade [11]. However, Avery and Zemsky [7] pointed out this difference. They defined an informational cascade as a situation where all the traders act ignoring their private information, whereas an agent is said to herd if, as a result of observing the actions of others, he makes a different choice from the one that he would make initially.

Starting from the nineties, several models of informational cascades have been proposed [7, 11, 126, 137, 28]. In most of these works, the basic principle which illustrates the occurring of an informational cascade is the following: consider a sequence of ex ante identical individuals who face similar choices, observe conditionally independent and identically distributed private information signals, and who observe the actions but

not the payoffs of predecessors. Suppose that individual i is in a cascade, and that later individuals understand this. Then individual $i + 1$, having gained no information by observing the choice of i , is, informationally, in a position identical to that of i . So $i + 1$ will also make the same choice regardless of his private signal. By induction, this reasoning extends to all later individuals, and the accumulation of information comes to a screeching halt once a cascade begins. Thus, the occurrence of an informational cascade translates into a *sudden transition* toward the same behavior.

In the following, we choose to present the details of the BHW model [15], which is considered the masterpiece among the models of informational cascades. Namely, most of the subsequent models are based on its mechanism. Then, some considerations on the limits of the existing models follows.

4.1 The BHW Model

The following are the main assumptions in BHW model for analyzing the onset of informational cascades.

- A number n of agents is considered. Each agent has to make a trading decision in a sequential order, which is an exogenous factor.
- All investors decide whether to invest or not to invest in an asset. The purpose for investment is to achieve the profit maximization.
- Each agent can observe the decisions of all those ahead of him, and make his own decision. There is a sequence of investment decisions by all the investor and the ordering is exogenous and is known to

all. There is no other form of information exchange among the traders.

- The agents obtain information by observing the investment decision ahead of them. This information is known as public information.
- The agents make their decisions in an uncertain situation, which means the investment profit V is uncertain when they decide to invest. Investment profit depends on the actual value of investment property, which is not known to investors when they make decisions.
- The two possible realizations of the investment properties are good situation G and bad situation B . Return on investment by the real value of investment properties as a good situation G is formulated as $V = 1$. When the investment property is in bad situation B , the investment return is formulated as $V = 0$.
- If the trader chooses to invest, it will result in some costs C . In the model $C = 0.5$, the costs are the same for all the investors. If the agent chooses not to invest, there will be no such costs.
- There is a prior probability of each situation (good situation and bad situation). The probability of the selection of investment in good situation and the probability of non-investment in bad situation is equal to 0.5.
- Each investor has his private signal (private information) about the actual value of investment, which is not observable by other investors. If the agents make investment decisions according to their own private signal, then this signal will be feedback into his

investment decisions. In good situation investors may invest, while in bad situation they may not invest. The amount of private signal has the characteristics of duality. Each agent's private signal X contains two values: H for the high precision (the possibility p is greater than 0.5), L for the low-precision (the probability is $1-p \leq 0.5$). Private signal can be transmitted through the investment behavior of investors. Namely, subsequent agents can infer the private signal of their predecessors by observing the trading actions of the latter. This means that, although transmittable, the information cannot be 100% transmitted. Therefore, the characteristics of information transmission meets with the conditions of $0.5 \leq p \leq 1$, which means that is very difficult for investors to rely only on their own private information to make accurate decisions.

The limitations of information's transmission require the investors to make a decision, based not only on private signal but also on the decisions of other investors. By using Bayesian theorem [115], the traders can make prior probability of real value of investment properties to posterior probability, and use the posterior probability to calculate the expected return on investment in order to make investment decision. Assume that γ is the posterior probability if the return of investment is $V = 1$. The expected investment income $E[V]$ is then

$$E[V] = \gamma \times 1 + (1 - \gamma) \times 0 = \gamma. \quad (4-1)$$

Expected return minus the investment cost C is the expected net profit P of the investment. Investment behavior can be divided into the following three types:

- 1) when $P > 0$, investor chooses to invest;
- 2) when $P = 0$, investor can choose to invest, or not to invest;

3) when $P > 0$, then investor chooses not to invest.

Through the analysis of the BHW model we can see that, when investors ignore their own signals and follow the decisions of the other investors, the informational cascade occurs. Also in Banerjee [11], which differs from the BHW model for the type of choice (the agents have to choose a number from an interval rather than having a binary choice), if two agents choose the same action, every subsequent agent will make the same choice, regardless of the signal he received. Imitation dominates private information.

To see how likely it is that a cascade occurs, consider the situation in which private signals are very noisy; specifically, the probability that the signal is correct is $p = 0.51$. A cascade occurs with slightly more than 75% after the first two players. After eight players, the probability that individuals are in a cascade increases up to 0.996. More generally, even when individuals have more accurate signals, the information contained in a cascade is not substantially better than a single individual's signal.

4.2 Considerations

The BHW and Banerjee models are, in their simplicity, very useful in the general understanding of some kinds of phenomena which can be observed in real world, such as fads, preference effects as in the choice of technologies, research topics, and eventually financial bubbles and crashes. In other words, they represent a useful stylized description of the process of social learning. However, the simplifying assumptions made in these models share some criticisms that need to be analyzed and revised.

First of all, these models are static models, thus they do not capture the dynamic adaptation and learning process of the agents.

Another aspect to point out is that, in a real investment context, the assumption that the timing and order of moves is exogenously given is unrealistic. Actually, when individuals have a choice of whether to delay, there can be long periods with no investment, followed by sudden spasms in which the action on one agent triggers the exercise of the investment option by many others. Moreover, the assumption of exogenous sequence generates what is called as “path dependence” [16]: the outcome of the cascade (good or bad cascade) strongly depends upon the sequence of movies.

Another implication of these models is the so-called “idiosyncrasy”: behavior resulting from signals of just the first few individuals drastically affects the behavior of all the subsequent followers. Of course, the idea that a few people are able to successfully influence the whole population is almost extreme (even the most influential dictators in history did not succeed to achieve this goal).

Of course, in reality we do not expect a cascade to last forever. Several possible kinds of shocks could dislodge a cascade: for example, the arrival of better informed individuals, the release of new public information, and shifts in the underlying value of adoption versus rejection. Indeed, when participants know that they are in a cascade, they also know that the cascade is based on little information relative to the information of private individuals. Hence, even after an informational cascade have persisted for a long time, it can be overturned with comparatively little effort. The alternation of fads is a clear example of how fragile a cascade could be.

Another fact to point out is the possible influence of the price dynamics on the onset of informational cascades. This aspect is not taken into account in [11, 15]. Conversely, in [7] the authors highlight that an informational cascade never takes place when prices adjust to reflect avail-

able information. Also Cipriani and Guarino [27] test experimentally herd behavior in asset markets with flexible prices, pointing out that the competitive price mechanism significantly reduced the occurrence of informational cascades. However, other experimental results seem to contradict this theory. In [43], for instance, the authors find that agents frequently acts against the market and their private information. Our suggestion is that the implications of such factor on informational cascades should be better analyzed. However, we have to point out that there exists a huge literature on the modeling of price dynamics, see [76] for a review, which shows that modeling price mechanism is not a simple task.

There is some doubt, eventually, as to whether these models have properly identified the usual source of difference in behavior across groups. Hence, they only consider the quite unrealistic case case of a *total* informational cascade, completely disregarding the possible differences in mass behavior across groups, or clusters, of agents. Actually, different groups may have different tendencies, different conversation patterns; the information may spread among the agents with different *intensities*, depending on the type and strength of interactions among the agents. In financial context, for instance, empirical evidence that herding phenomena do not involve all agents at the same time abounds [70, 127].

Summing up, the classical models of informational cascades share some unrealistic assumptions that need to be complemented in such way. Our aim is that of approaching to this phenomenon from a different perspective: we will focus on informational cascades from a control viewpoint. We will propose a dynamic model which allows to generate informational cascades of different intensities. This approach will allow us to exploit our background in the field of control theory in order to overcome some

of the limitations of the aforementioned models, such as path dependence, or the unrealistic case of total informational cascades. In the next chapters, we extensively explain the details and the advantages of our approach.

A New Model of Opinion Dynamics

Models of social networks are structures made up of individuals that are tied based on their interdependency. Such models explain the confidence or influence flow in populations without relying on detailed social psychological findings. The process of opinion dynamics evolves along the networks of social influence and affects the structure of the network itself. In the field of social networks, opinion dynamics is of high interest in many areas including politics, as in voting prediction; physics, as in spinning particles; sociology, as in the diffusion of innovation, the electronic exchange of personal information, and language change; and finally economics. That's why the study of opinion dynamics has recently started to attract also the attention of the control community, whose main challenge is represented by the analysis of the stability properties of the proposed models, in particular, convergence of the agents' opinions, as shown in some of the most recent contributions in this field [56, 63, 101, 107].

5.1 A Brief Survey on Opinion Dynamics

In the literature, two general lines of research have been proposed in order to explain the dynamics of opinion consensus or disagreement. The first one includes many models of opinion dynamics based on “bounded confidence”, which means that an individual only interacts with those whose opinions are close enough to its own. This idea reflects the psychological concept called *selective exposure*. Broadly defined, “selective exposure refers to behaviors that bring the communication content within reach of one’s sensory apparatus” [139]. The other line of research focuses on the “obstinacy” of agents: in general, an agent neither simply shares nor strictly disregards the opinion of any other agent, but takes into account the opinions of others to a certain extent in forming his own opinion.

In what follows, we briefly overview these two lines of research, starting from the respective pioneer works.

5.1.1 Bounded Confidence Models

Recently, bounded confidence (BC) models of opinion dynamics, a label coined by Krause in 1998 [69], have received significant attention. BC models are models of continuous opinion dynamics in which agents have bounded confidence in others opinions. The first version of BC models was formulated by Hegselmann and Krause [63], called the HK model, where agents synchronously update their opinions by averaging all opinions in their confidence bound:

$$\dot{x}_i(t) = \sum_{j: |x_j(t) - x_i(t)| < d} (x_j(t) - x_i(t)), \quad (5-1)$$

with $d > 0$.

The other popular version of BC models was developed and investigated by Deffuant and Weisbuch [125], called the DW model. The HK and DW models are very similar, but they differ in their update rule: in the DW model a pairwise-sequential updating procedure is employed instead of the synchronized one. In the HK model, the set of neighbors of an agent is defined as those agents whose opinions differ from his opinion by less than a confidence bound. Hence, this model is dealing with endogenously changing topologies, that is, state dependent or changing from inside, in contrast to the exogenously changing topologies. The HK models are classified based on various factors: a model is called agent- or density-based if its number of agents is finite or infinite, respectively, and a model is called homogeneous or heterogeneous if its confidence bounds are uniform or agent-dependent, respectively. The convergence of both agent- and density-based homogeneous HK models are discussed in [17], while the agent-based homogeneous HK system is proved to reach a fixed state in finite time [41]. Based on this model of opinion dynamics, which main feature is that of considering a state-dependent topology, several interesting works followed, see for instance [89, 22, 133, 117].

5.1.2 Models of Opinion Pooling

One of the first who analyzed consensus of individuals' opinion was De-Groot [34], who proposed Eq. (2-8) as a model of opinion dynamics. It is actually an iterative scheme of opinion pooling in which each agent updates his opinion based on his own and neighbors' current opinion. This model was extended by Friedkin and Johnsen (FJ) [55], who introduced the concept of agents' *susceptibility* to interpersonal influence, and also provided an estimation of the agents' opinion at equilibrium. FJ model in matrix form is described as

$$x(k+1) = \Lambda W x(k) + (I_N - \Lambda)x(0), \quad (5-2)$$

where W is a row stochastic matrix of interpersonal influences, and $0 \leq \Lambda \leq I_N$ is a diagonal matrix of agents' susceptibilities to the social influence.

The natural and intensively investigated special case of this model assumes the coupling condition $\lambda_{ii} = 1 - w_{ii} \forall i$, that is, $\Lambda = I_N - \text{diag}(W)$. Under this assumption, the selfweight w_{ii} plays a special role, considered to be a measure of stubbornness or closure of the i -th agent to interpersonal influence. If $w_{ii} = 1$, and thus $w_{ij} = 0 \forall j \neq i$, then he is maximally stubborn and completely ignores opinions of his neighbors. Conversely, if $w_{ii} = 0$, and thus his susceptibility is maximal ($\lambda_{ii} = 1$), the agent is completely open to interpersonal influence, attaching no weight to his own opinion. Thus, the susceptibility of the i -th agent λ_{ii} varies between 0 and 1, where the extremal values correspond to maximally stubborn and open-minded agents, respectively.

Starting from this simple model of opinion dynamics, several models followed. Some of them investigate the presence of leaders in the population, and their effects on opinion consensus [68, 85, 90, 118, 58]. Another interesting framework concentrates on a class of randomized dynamics [108, 54, 49, 109, 112]. In [54], for instance, the authors propose a model inspired to the FJ model where nodes interact in randomly chosen pairs, following the so called gossip protocol [18]. At each time step, a randomly chosen agent updates its opinion to a convex combination of its own opinion, the opinion of one of its neighbors, and its own initial opinion or “prejudice”. They show that the dynamics persistently oscillates; however, the result is a stable opinion profile on average. This stability property guarantees that the dynamics, although affected by persistent random oscillations, possesses an ergodic behavior. For an overview on randomized algorithms for opinion formation, see [53] and the references therein. Another interesting concept in models of opinion dynamics as in

(2-8) is that of democratic consensus, which is introduced and discussed in [48].

One of the limitations of most of the models of opinion dynamics are the strong assumptions on the graph of interactions required to reach consensus. Just as an example, in [88] the authors study consensus of a multi-agent system with cooperative-antagonistic interactions and switching topologies in a discrete-time setting. Both unidirectional and bidirectional topologies are considered. It was proven that the limits of all agent states exist and reach a consensus only if the topology is uniformly jointly strongly connected or infinitely jointly connected. Of course, in real applications these assumptions are quite unrealistic.

However, an interesting benefit of the FJ model is that, despite being a simple model, it is actually a very flexible model which may be adopted in various contexts. In our artificial market, we have translated the agents' interaction as a variation of the risk attitude due to the neighbors' influence. This interaction mechanism has been described by Eq. (3-10), which is actually equivalent to the FJ model of opinion dynamics (5-2). Indeed, the risk attitude in its general meaning can be seen as a human dynamical feature which may be affected by different factors. Undoubtedly, there is an innate predisposition which characterizes each person, and which depends on the character, education, life experience, and so on. Moreover, people's attitude toward risk is inevitably influenced by exogenous and environmental factors, and overall by the interaction with other people. That's why we leverage the FJ model to capture the evolution of the agents' risk attitude, which is updated taking into account both their innate predisposition, described by $y(0)$, and the interaction with other people, described in the matrix of interpersonal influences W .

5.2 The Model

From now on, we introduce the main contributions of this thesis. In order to study informational cascades from a control viewpoint, we see this phenomenon as a result of a diffusion of a certain opinion in an ensemble of agents. To do this, we propose a new model of opinion dynamics which allows us to see informational cascades as a consensus problem. In this model, we will try to embed some of the main features of herding and of the consequent informational cascades, and, at the same time, to overcome some limitations of the models of informational cascades already present in the literature. As already pointed out in Section 4.2, one of the limitations of these models lies in the fact that eventually all the agents are involved in the triggered informational cascade. Undoubtedly, this perspective is almost unrealistic, as shown by empirical evidence. These observations lead us to consider the more realistic case of “partial informational cascade” as the partial diffusion of a common opinion in an ensemble of agents.

Differently from the classical models of informational cascades, which are static models based on bayesian rules, we propose a dynamic networked model in which each agent interact with some other agents, and this interaction biases his opinion.

Exploiting the well established link between multi-agent systems and graph theory [13], and to be compliant with the models of opinion dynamics, we see each trader as a node of a dynamical network and use the network edges to model the influence among the agents.

In our model, the state of each node i is described by two variables, $x_i(k)$ and $r_i(k)$. The former captures the opinion of agent i , while the latter his reputation. We model the interaction and the mutual influence among the agents as a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \{\alpha_{ij}(k)\}_{i,j \in \mathcal{V}})$, where each

node i of the set \mathcal{V} represents an agent, and the generic edge e_{ij} is encompassed in the edge set \mathcal{E} if the opinion of node i depends on that of node j . Moreover, the influence of the opinion of the other agents on the opinion of agent i is weighted through the coefficient

$$\alpha_{ij}(r(k)) = \frac{r_j(k)}{\sum_{h \in \mathcal{N}_i} r_h(k)}, \quad (5-3)$$

where \mathcal{N}_i defines the set of in-neighbors of i (see Def. 2.1.4).

We consider the following dynamics for the state variables $x_i(k)$ and $r_i(k)$:

$$\begin{cases} x_i(k+1) = x_i(k) + \sum_{j \in \mathcal{N}_i} \alpha_{ij}(r(k)) (x_j(k) - x_i(k)), \\ r_i(k+1) = f(r_i(k), u_i(k), v_i(k)). \end{cases} \quad (5-4)$$

We model through $f(r_i(k), u_i(k), v_i(k))$ the dependence of an agent's reputation from his actions $u_i(k)$ and exogenous factors $v_i(k)$, constraining the vector field f to describe a positive system to ensure the reputation of an agent be positive.

By doing so, the dynamics of $x_i(k)$ becomes that of a directed network of diffusively coupled integrators with state dependent gains. We can leverage the topology of this network to represent people with different opinion formation schemes. Namely, we model the agents who are not willing to let their opinion be influenced by that of the other agents as the set $\mathcal{R} := \{i : \mathcal{N}_i = \emptyset\}$ of nodes with zero indegree. The nodes of \mathcal{R} , having outgoing edges, play the role of leaders, as they are capable of influencing the opinion of other people. On the contrary, nodes with high indegree may represent the influencees, that is, individuals who are subject to the influence of the mass, thus uniforming their ideas and opinions with those of the majority of the population. Finally, nodes with low, but nonzero indegree model agents who, conscious of their lack of experience and information, follow the opinion of a small set of peers

which they consider experts and clever. Consistently, the outdegree measures the potential of an individual of influencing the population, while his actual ability of influencing his peers is related to his reputation $r_i(k)$ through the coefficient $\alpha_{ij}(r(k))$ in Eq. (5-3). The larger $r_i(k)$, the stronger the influence that i exerts on the nodes connected to him. In this way, the agents' reputation becomes a measure of the influence that the agents exert on their peers, and thus we capture from a dynamical viewpoint the tendency of people to follow the actions of individuals with high reputation, which is considered one of the main causes of herding behavior [39].

We emphasize that this model is a general model of opinion dynamics which can be embed in different contexts and situations where an ensemble of individuals have to reach a consensus on a certain opinion (e.g. voting). In the next chapter, we will use this model in a financial context, showing its ability in reproducing herding phenomena.

Informational Cascade as a Pinning Control Problem

In what follows, we show how the proposed model of opinion dynamics succeeds in reproducing informational cascades of different intensities. First, we show how the model allows to view informational cascades as a pinning control problem. Then, exploiting tools from partial pinning control (Sec. 2.4), we predict the magnitude of the triggered informational cascades. We propose a numerical application to test the effectiveness of our predictions.

To reproduce partial informational cascades, we populate network (5-4) with two categories of agents: *informed* and *followers*. These qualitatively correspond to the extreme types of actors in the FJ model (5-2). Informed agents are stubborn and thus their opinion dynamics is not influenced by other people; conversely, the followers completely disregard their opinion and only take into account that of their neighbors. To be compliant with the classical models of informational cascades, we assume the presence of an external information injected on a subset of the network nodes. We model the exogenous information as an exter-

nal signal \bar{x} fed by an additional node only to a subset of the agents, say $\mathcal{I} \subseteq \mathcal{R}$, which then play the role of informed agents. From these influencers, the information diffuses through the network, affecting the agents' decision making. This choice allows us to see herding dynamics as a *pinning control problem*, with the additional node playing the role of the *pinner*, and the informed nodes being in the role of *pinned nodes*. Consistently, Eq. (5-4) may be rewritten as

$$\begin{cases} x_i(k+1) = x_i(k) + \sum_{j \in \mathcal{N}_i} \alpha_{ij}(r(k))(x_j(k) - x_i(k)) \\ \quad + \delta_i(\bar{x} - x_i(k)), \\ r_i(k+1) = f(r_i(k), u_i(k), v_i(k)), \end{cases} \quad (6-1)$$

where $\delta_i = 1$ for all $i \in \mathcal{I}$, and $\delta_i = 0$ for all $i \notin \mathcal{I}$.

By constraining \mathcal{I} to be a subset of \mathcal{R} , the informed agents will only be influenced by the exogenous signal \bar{x} , thus disregarding their own beliefs and the opinion of the other investors.

To model the effect of each agent's opinion on his behavior, we define the agent decision as a generic output $s_i(k)$ of system (6-1):

$$s_i(k) = g(x_i(k)). \quad (6-2)$$

The function g , of course, may be explicated depending on the context. For the informed influencers, we will have that $s_i(k) = g(\bar{x}) = \bar{s}$, $\forall i \in \mathcal{I}$, and $\forall k > 0$. As Eq. (6-1) is that of a diffusive process over a directed network of discrete integrators, it may well be the case that the opinion of an additional set of nodes converges to that of the informed agents, thus ensuring that also the decisions $s_i(k)$ of the former replicate that of the latter, that is, \bar{s} . The larger the set of nodes \mathcal{C} which reaches consensus to the opinion of the informed agents, the larger the magnitude of the herding phenomenon that is generated. Hence, to understand the ability of the model in eqs. (6-1)-(6-2) of generating informational cascades of

different magnitudes, we shall study the variation of the number of nodes whose opinion converges to that of the informed agents, in dependence of the selection of the set \mathcal{I} . Note that, as $\mathcal{I} \subseteq \mathcal{C}$, then $\mathcal{C} \neq \emptyset$.

Unfortunately, as already pointed out in Sec. 2.3, while the literature on pinning control is extremely vast, no results hold for dynamics like that in Eq. (6-1), where $x_i(k)$ evolves according to a discrete-time and state dependent law. Hence, no results in the literature provide tools to accurately predict our diffusion process. Still, given the set of pinned nodes \mathcal{I} , we can rely on the topological conditions available in the literature to predict the number of nodes which *should* reach consensus to the pinner's opinion. In particular, we use the pinned node selection algorithm proposed in [37] (see Section 2.4.1), which translates the problem of maximizing the number of pinning controlled nodes into an optimization problem on a graph. To do so, it relies on the following structural condition: a node is pinning controllable if all of the RSCCs (see Def. 2.1.16) in its upstream encompass at least a pinned node. Hence, the algorithm selects the set of p RSCCs to be pinned that maximize the number of nodes fulfilling such condition. To allow numerical solvability, this graph optimization is translated into an integer linear program. Then, we numerically verify if, considering the dynamics in Eq. (6-1) which does not fulfill the hypothesis of partial pinning control, yields results that are consistent with these structural conditions. To do this, we test our model of opinion dynamics in the artificial market proposed in Chapter 3. We show that, by selecting the nodes of the set \mathcal{I} according to the partial pinning control algorithm, our opinion dynamics model succeeds in generating herding phenomena of different and predictable intensities in our artificial financial market.

In what follows, we propose the financial application of our model of opinion dynamics.

6.1 An Application to our Artificial Financial Market

The artificial market proposed in Section 3.1 is populated by N agents, and we select a directed weighted graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \{\alpha_{ij}(k)\}_{i,j \in \mathcal{V}})$ to model the mutual influence among them. Notice that, while the network structure is fixed, (\mathcal{G} and \mathcal{E} are time-invariant), the interaction weights are time-varying. This assumption is reasonable if we think that we are dealing with informational cascades, which are phenomena occurring on a short time scale. Thus, we can assume that in a brief period of time the pattern of relationships of each agent is keeping fixed, even though the intensity of such relations may change.

At each time step $k = 1, \dots, T$, the agents, in a sequential random order, can trade in a set of financial assets \mathcal{L} , with $|\mathcal{L}| = L$. Notice that T is selected high enough so that, on average, no one is favored by the trading sequence. The assets are characterized by stochastic returns $\beta = [\beta^1, \dots, \beta^L]$ and by a limited availability. We denote with $\bar{x} = E[\beta]$ the expected return of the assets, which is assumed to be constant¹, and relabel the assets in \mathcal{L} such that $\bar{x}^1 > \bar{x}^2 > \dots > \bar{x}^L$.

In line with the proposed model of opinion dynamics (Eq. (6-1)), the state of each trader i is described by two state variables. As we are in a financial context, we assume that the vector $x_i(k) = [x_i^1(k), \dots, x_i^L(k)]$ represents the current evaluation of the asset returns made by agent i , while the scalar $r_i(k)$ quantifies his reputation. It is reasonable to assume that the latter coincides with the agent's current wealth, whose

¹As already pointed out in Section 4.2, we are aware that price dynamics is an important market feature which may influence the onset of cascades. However, much effort should be spent in order to build a model of price mechanism. That's why in this study we will not take into account this issue. However, this could be object of further research.

dynamics is described in Eqs. (3-5) - (3-6): the richer the agent, the more prone his neighbors are to take into account his opinion.

As to the first state variable, notice that, at a first instance, we had assumed that all the agents have the correct information on the assets' return. From now on, we remove this hypothesis, assuming that each agent has a personal evaluation on the assets, and thus associates different probability distributions to the assets' return. This more realistic behavior allows to be compliant with the classical models of informational cascades, in which the opinion of an investor is affected by exogenous information. Actually, in this context, this information may represent the correct evaluation of the expected asset returns \bar{x} fed by a virtual trader, the pinner, only to the subset \mathcal{I} of influencers, the so called *informed* traders.

As in Section 3.1.2, the agents determine their trading strategy maximizing their expected utility. Without loss of generality, we now populate our market of risk neutral investors [64, 66]. From a mathematical point of view, this means setting the risk attitude of each agent $y_i(k) = 1, \forall i, k$ in Eq. (3-1). In the view of this, the expected utility that agent i associates to the assets becomes:

$$E[U_i(k)] = \epsilon r_i(k) x_i(k), \quad (6-3)$$

with $x_i(k) = 0.5(a_i(k) + b_i(k))$ being the current evaluation of the asset returns made by agent i , and ϵ being the fraction of current wealth that each agent is prone to invest. Accordingly, the trading preferences $s_i(k)$ of agent i in (3-3) become an output of $x_i(k)$, in line with Eq. (6-2). Namely, the opinion of each agent on the expected returns of the assets reflects in his trading strategy. Consistently, (3-4) becomes

$$s_i^j(k) := m : \exists m-1 \ x_i^l(k) > x_i^j(k), \exists L-m \ x_i^l(k) < x_i^j(k). \quad (6-4)$$

Due to the limited availability of the assets, the actual trading action made by agent i corresponds to trade in the first available asset l^* according to his preferences $s_i(k)$, and thus the outcome of the trade is a realization of β^{l^*} .

In the view of these considerations, Eq. (6-1) can be rewritten as

$$\begin{cases} x_i(k+1) = x_i(k) + \sum_{j \in \mathcal{N}_i} \alpha_{ij}(r(k))(x_j(k) - x_i(k)) \\ \quad + \delta_i(\bar{x} - x_i(k)), \\ r_i(k+1) = f(r_i(k), \beta^{l^*}, \tau(k)), \end{cases} \quad (6-5)$$

where $\tau(k)$ is the exogenous factor representing the selected taxation scheme which affects the current wealth (see Eq. (3-6)).

6.1.1 Herding Intensity

To test the ability of the proposed model of triggering herding phenomena of different magnitudes, we perform a set of numerical simulations in which we vary the number of pinned nodes $p = |\mathcal{I}|$. We vary p between one and the minimal value of influencers required to generate an informational cascade that involves all the agents. In each simulation, say the p -th, the set $\mathcal{I}(p)$ is selected according to the pinned node selection algorithm proposed in [37]. On the basis of topological conditions, this algorithm maximizes the cardinality $|\mathcal{C}(p)|$ of the set $\mathcal{C}(p)$ of nodes which should reach consensus to the pinner's value. We stress that, as the dynamics of $x_i(k)$ in Eq. (6-5) do not fulfill the assumptions made in [37], the set $\mathcal{C}^o(p)$ of nodes that will actually achieve consensus on the state of the pinner in the p -th simulation could be different from the set $\mathcal{C}(p)$.

We recall that, for the informed traders, $x_i(k) = \bar{x}$, and thus $s_i(k) = \bar{s} = [1, \dots, L] \ \forall i \in \mathcal{I}$, and $\forall k > 0$. On the other hand, the opinion, and consequently the trading action, of the influencees depends on that

of their peers. We say that an influencee herds when he uniform his trading strategy to that of the informed agents. For each simulation p , the set of the agents involved in the triggered informational cascade from a certain time step k^* is then defined as

$$\mathcal{H}(p) := \{i \in \mathcal{V} : s_i(k) = \bar{s}, \forall k > k^*\}. \quad (6-6)$$

Thus, $\mathcal{H}(p)$ is the observed set of nodes the trading strategy of which coincides with that of the pinner in the last $T - k^*$ iterations.

For each simulation, we measure the magnitude of the triggered informational cascade through the LSV index [72], a well-established measure of the strength of herding phenomena. It is defined as

$$LSV(p) = \left| \frac{|\mathcal{H}(p)|}{N} - b \right|, \quad (6-7)$$

where b refers to the fraction of agents who have correct preferences of the assets in the no-herding case, that is, when $p = 0$.

6.2 Simulations' Setting

In order to select an appropriate topology of interactions for this context, one could try to reconstruct the network starting from epirical data. However, this is a well known hard task, and for the moment is out of our scope. That's why we performed numerical simulations on several real network topologies, as we want to highlight the robustness of our model to any topology of interactions. In what follows, we extensively illustrate the case of a selected real network topology. However, later on we will briefly show some evidence of other topologies, in order to prove that in every case we have obtained the same qualitative results. In line with the selected network, we consider a market populated by $N = 1057$ agents endowed with the same initial wealth $r_i(0) = 100\$$, $\forall i = 1, \dots, N$.

At each iteration $k = 1, \dots, T = 5000$, each agent is prone to trade the fraction $\epsilon = 0.2$ of his current wealth $r_i(k)$ in one of the $L = 4$ categories of financial assets. Each category is characterized by an availability equal to 1/15th of the total wealth of the system, except for the fourth asset, which is a virtual asset and correspond to no-investment, and thus has an unlimited availability. In line with our scenario, the expected returns of the assets are $\bar{x}^1 = 1.065$, $\bar{x}^2 = 1.050$, $\bar{x}^3 = 1.035$, and $\bar{x}^4 = 1$. All the agents have a distorted initial perception of the expected returns (except for the virtual asset), that is, $x_i^l(0) = \bar{x}^{L-l} \forall i = 1, \dots, N$ and $\forall l = 1, 2, 3$. For the sake of simplicity, we consider only a taxation scheme in our simulations, that is, the Tobin-like tax (see Section 3.1.3.1). We evaluate the onset of the herding phenomenon in the last 1000 iterations ($k^* = 4000$).

6.3 Main Results

Given these premises, we can now go through our numerical results. We start by noting that, although the dynamics of $x_i(k)$ in Eq. (6-5) do not fulfill the assumptions made in [37], we observe that $|\mathcal{C}(p)| = |\mathcal{C}^o(p)| \forall p$. As the opinion of all agents in the set $\mathcal{C}^o(p)$ converges to that of the pinner, i.e. $x_i(k) = \bar{x} \forall i \in \mathcal{C}^o(p)$, and $\forall k = T - 1000, \dots, T$, we also have that $s_i(k) = \bar{s} \forall i \in \mathcal{C}^o(p)$ and $\forall k = 1, \dots, T$. In other words, all agents achieving consensus on the state of the pinner also imitate the latter's trading strategy. Hence, the set $\mathcal{C}(p)$ represents a prediction of the set $\mathcal{H}(p)$ of agents who herd in the p -th simulation. As, when $p = 0$, that is, in the no-herding case, $x_i^l(k) = \bar{x}^{L-l} \forall l = 1, 2, 3$, and $\forall i, k$, we have that $b = 0$, and thus Eq. (6-7) becomes

$$LSV(p) = \frac{|\mathcal{H}(p)|}{N}. \quad (6-8)$$

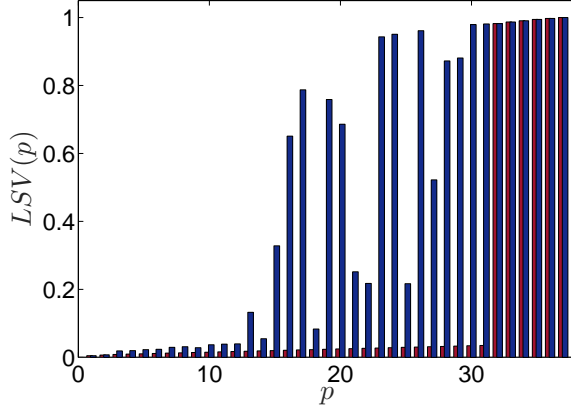


Figure 6-1: Intensity of the predicted (red bars) and observed (blue bars) informational cascade for each value of p .

This allows to predict the herding intensity as

$$\widehat{LSV}(p) = \frac{|\mathcal{C}(p)|}{N}. \quad (6-9)$$

As shown in Fig. 6-1, we find that $LSV(p) \geq \widehat{LSV}(p) \forall p$, as we observe that the set of herding agents is often larger than the set $\mathcal{C}(p)$, that is, $\mathcal{C}(p) \subseteq \mathcal{H}(p)$.

By inspecting the opinion dynamics of the agents in the simulations for which $LSV(p) > \widehat{LSV}(p)$, we notice that the agents in the set $\mathcal{H}(p) - \mathcal{C}(p)$ perform the correct ranking of the assets although they do not reach consensus on the state of the pinner, i.e., $\forall i \in \mathcal{H}(p) - \mathcal{C}(p)$, we have

$$\begin{cases} x_i(k) \neq \bar{x}(k); \\ x_i^1(k) > x_i^2(k) > x_i^3(k) \quad \forall k = T - 1000, \dots, T, \end{cases} \quad (6-10)$$

yielding $s_i(k) = \bar{s} \forall i \in \mathcal{H}(p) - \mathcal{C}(p)$. This because, as shown in Fig. 6-2, the expected returns perceived by the agents in the set $\mathcal{H}(p) - \mathcal{C}(p)$ are close enough to those of the pinner to determine the same output, that is,

the same trading strategy. These influencees, being in the downstream of several influencers but also of some non informed leaders, feel the opinion of the latter as a disturbance on their opinion, thus compromising their ability of converging to the correct expectations.

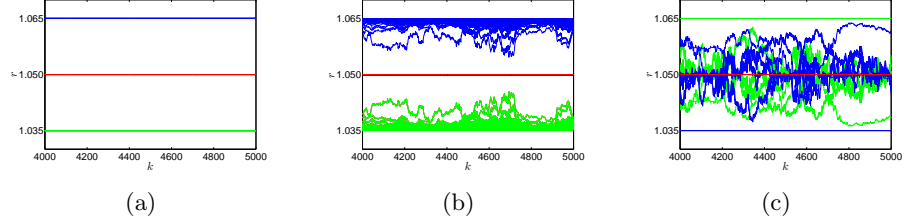


Figure 6-2: Expected returns of the asset 1 (blue lines), 2 (red lines), and 3 (green lines) perceived by the agents included in the set $\mathcal{C}(p)$ (Fig. 6.4(a)), $\mathcal{H}(p) - \mathcal{C}(p)$ (Fig. 6.4(b)), and $\mathcal{V} - \mathcal{H}(p)$ (Fig. 6.4(c)), respectively, for the simulation $p = 31$.

6.4 Further Results

In the following, we will show some further results, in order to highlight the robustness of the proposed model to variations of initial conditions, and of the selected topology of interaction.

6.4.1 Initial Conditions

In our simulations, we have intentionally maximized the distortion of the initial perception of the expected returns of the agents to better highlight the emergence of herding behavior. Namely, we show how the agents involved in the informational cascade drastically change their opinion even though their initial opinions are opposite to those of the pinner. However, a random selection of the initial opinions does not qualitatively

affect our results. Even in this case, indeed, we observe that the set $\mathcal{C}(p)$ represents a prediction of the minimum number of agents that actually herd. Namely, we have that $|\mathcal{C}(p)| = |\mathcal{C}^o(p)|$, and that $\mathcal{C}(p) \subseteq \mathcal{H}(p) \forall p$, as shown in Fig. 6-3. In the view of this, we can assess that our model is independent from the initial conditions, thus overcoming one of the limits of the classical models of informational cascades (see Section 4.2).

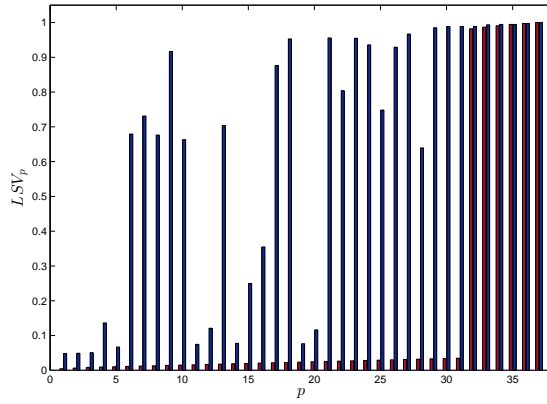


Figure 6-3: Intensity of the predicted (red bars) and observed (blue bars) informational cascade for each value of p , and for a random selection of the agents' initial opinions.

6.4.2 Topologies of Interaction

As already pointed out, in our simulations we selected a real sufficiently large network. Actually, we performed numerical simulations with different networks of interaction, obtaining the same qualitative results. Indeed, for each scenario of interaction, we have that $|\mathcal{C}(p)| = |\mathcal{C}^o(p)| \forall p$, while in Fig. 6-4 we can observe the triggered partial informational cascades for three of these scenarios.

These further results share the robustness of our model to variations of initial conditions, and network interactions, thus highlighting the flexibility of our model to be adapted to different contexts.

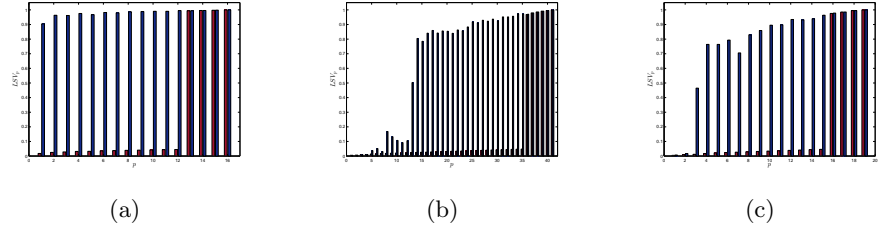


Figure 6-4: Intensity of the predicted (red bars) and observed (blue bars) informational cascade for each value of p , and for three different networks of interaction.

Phase Transitions in Partial Pinning Control of Complex Networks: the Generating Functions' Approach

7.1 About Phase Transitions

The term phase transition (or phase change) is most commonly used to describe transitions between solid, liquid and gaseous states of matter, and, in rare cases, plasma (physics). A phase of a thermodynamic system and the states of matter have uniform physical properties. During a phase transition certain properties of the system change, often discontinuously, as a result of the change of some external condition, such as temperature, pressure, or others. For example, a liquid may become gas upon heating to the boiling point, resulting in an abrupt change in volume. The measurement of the external conditions at which the

transformation occurs is termed phase transition. Phase transitions are common in nature and used today in many technologies.

The condition for phase transitions generally stems from the interactions of a large number of particles in a system, and does not appear in systems that are too small. Phase transitions can occur and are defined for non-thermodynamic systems, where temperature is not a parameter. Examples include: quantum phase transitions, dynamic phase transitions, and topological (structural) phase transitions. In these types of systems other parameters take the place of temperature. For instance, connection probability replaces temperature for percolating networks.

Paul Ehrenfest [67] classified phase transitions based on the behavior of the thermodynamic free energy E as a function of other thermodynamic variables, such as temperature T . Under this scheme, phase transitions were labeled by the lowest derivative of the free energy that is discontinuous at the transition. The transitions are classified as

- first-order phase transitions, which exhibit a discontinuity in the first derivative of the free energy with respect to some thermodynamic variables. First-order phase transitions are those that involve a latent heat. During such a transition, a system either absorbs or releases a fixed (and typically large) amount of energy per volume. During this process, the temperature of the system will stay constant as heat is added.
- second-order phase transitions, which are continuous in the first derivative (the order parameter, which is the first derivative of the free energy with respect to the external field, is continuous across the transition), but exhibit discontinuity in a second derivative of the free energy.

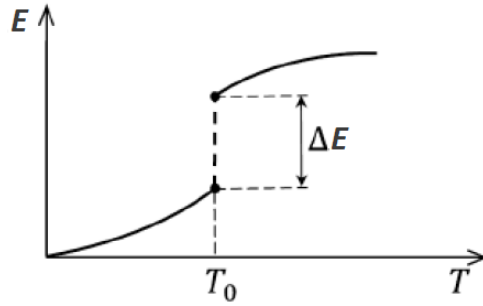


Figure 7-1: An example of first order phase transition.

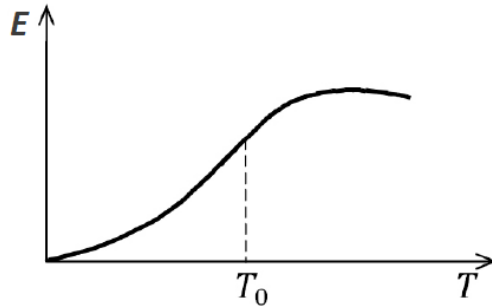


Figure 7-2: An example of second order phase transition.

7.1.1 Phase Transitions in Partial Pinning Control

Pinning control in complex networks is a synchronization technique around a desired trajectory $s(k)$. Applying partial pinning control on a given network, we actually observe a transition from a fully incoherent behavior to the synchronization of the whole network. This phenomenon may be associated to a phase transition, in which the coherence parameter is represented by the fraction of pinning controllable nodes.

Namely, let's give a deeper look to Fig. 6-1, focusing our attention

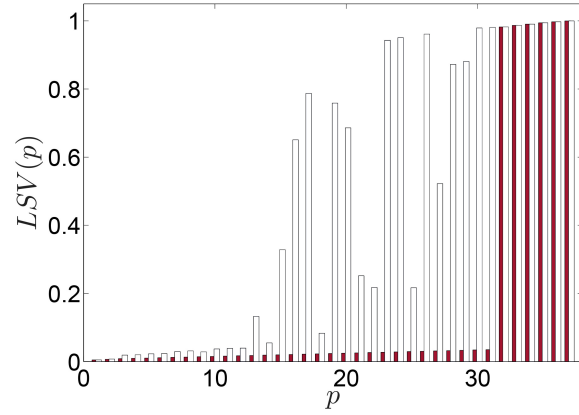


Figure 7-3: Fraction of pinning controllable nodes for each value of p .

on the intensity of informational cascade predicted through the partial pinning control algorithm, see Fig. 7-3. As we can see, by increasing the number of appropriately selected pinned nodes, we can notice a first order phase transition from a fully incoherent network behavior to the synchronization of the whole network to the pinner's trajectory.

Abstracting from the application proposed in the previous chapter, Fig. 7-3 represents an analysis of partial pinning controllability of the selected network. Namely, the analysis of the structural conditions for partial pinning controllability of a generic network provides lots of information on the network itself, such as the minimum number of pinned nodes required in order to control the whole network, how many nodes can be controlled with a limited amount of resources, represented by the pinned nodes, and how much effort must be done in order to control almost the whole network.

By inspecting Fig. 7-3 we have realized that the emergence of a first order phase transition depends on the existence of a Giant Strongly

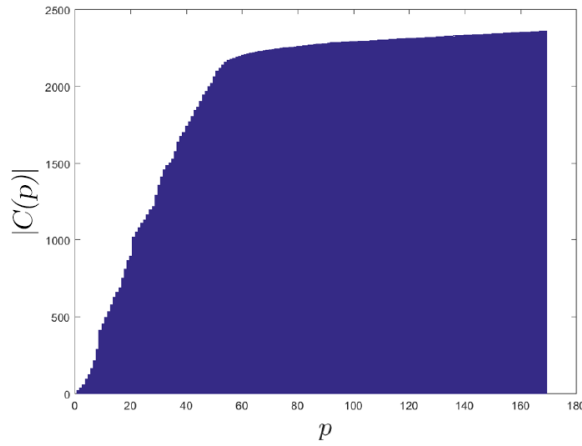


Figure 7-4: Partial pinning controllability of a network of 2400 nodes which does not include a GSCC. Notice the emergence of a second order phase transition.

Connected Component (GSCC, see Section 2.1.3.1) in the network. The GSCC behaves like latent heat and represents the number of nodes to be pinning controlled to get the phase transition and p^* is the minimum number of Root Strongly Connected Components (RSCC, see Section 2.1.3.1) to be pinned to pinning control the GSCC. Actually, networks without a GSCC exhibit a second order phase transition. An example is given in Fig. 7-4. Thus, the type of the emerging phase transition in partial pinning controllability of a complex network is strictly related to the topological features of the latter, most of them are well explained by the degree distribution of the network itself.

In the view of this, a question naturally arise. Are we able to analytically predict the partial pinning controllability of a given network, without leveraging the numerical simulations? In particular, some issues could be addressed, such as:

Questions 7.1.1.

- 1) *Starting by the only knowledge of the degree distribution of a given network, can we analytically predict which kind of phase transition will arise?*
- 2) *As to the first order phase transition, can we estimate the width of the discontinuity, in order to predict the number of controllable nodes in correspondence of the transition?*
- 3) *Can we predict the minimum number of nodes which should be pinned in order to observe the jump?*

To answer this questions, a statistical mechanics approach could be considered. In particular, we will try to leverage the generating functions to better characterize the phenomenon of phase transition in partial pinning controllability of complex networks. The details of this approach are extensively explained in the following.

7.2 Statistical Mechanics of Complex Networks

The study of network models began with Erdős and Rényi [45, 46, 47]. They proposed a model of networks with randomly distributed links. The random graph of Erdős and Rényi is one of the most studied models of a network, and possesses the considerable advantage of being exactly solvable for many of its average properties. However, due to the development of computers, allowing the analysis of large amounts of data, such as the Internet and WWW, some analysis of real world networks has been done in the last decade. In particular, other models of complex networks were proposed in order to capture some features of real networks, such as the well known scale-free and small-world networks. Although these models are more appropriate than the random network

to represent some real networks, they still differs from real-world topologies in some fundamental ways, as pointed out in [116, 3].

In the view of this, recent works on the structure of complex networks have focused attention on graphs with arbitrary degree distributions. Studying the properties of random graphs defined by their degree distribution is not simply an abstract problem; it has a clear practical motivation. For instance, one may consider an empirical degree distribution that is based on measured or observed data, and try to extract information on the network itself which do not follow a specific degree distribution.

Clearly, a degree distribution does not define a graph uniquely. That said, an attractive alternative to the classical models is to define a random graph by a given degree distribution assuming that apart from the degree distribution the graph is absolutely random. This line of research was introduced by Molloy and Reed [91] and was later developed further by Newman, Strogatz, and Watts, see [96, 95], whose main results are summarized in the following.

7.2.1 Generating Functions of Undirected Networks

Newman, Strogatz, and Watts developed a formalism for calculating a variety of quantities, both local and global, on large graphs with arbitrary probability distribution of the degrees of their vertices, in the thermodynamic field. In all respects other than their degree distribution, these graphs are assumed to be entirely random. This means that the degrees of all vertices are independent identically distributed random integers drawn from a specified distribution. For a given choice of these degrees, also called the “degree sequence”, the graph is chosen uniformly at random from the set of all graphs with that degree sequence.

Their approach is based on the generating functions [130], the most fundamental of which is the generating function $G_0(x)$ for the probability distribution of vertex degrees k , which is defined as

$$G_0(x) = \sum_{k=0}^{\infty} p_k x^k, \quad (7-1)$$

where p_k is the probability that a randomly chosen vertex has degree k , and $|x| \leq 1$. As p_k is assumed to be correctly normalized, we have that

$$G_0(1) = 1. \quad (7-2)$$

The probability distribution p_k is given by the k^{th} derivative of $G_0(x)$:

$$p_k = \frac{1}{k!} \left. \frac{d^k G_0}{dx^k} \right|_{x=0}. \quad (7-3)$$

Thus, the function $G_0(x)$ *generates* the probability distribution p_k .

Consequently, the average degree z of a network is given by

$$z = \langle k \rangle = \sum_k k p_k = G'_0(1), \quad (7-4)$$

while higher moments of the distribution can be computed from higher derivatives:

$$\langle k^n \rangle = \sum_k k^n p_k = \left[\left(x \frac{d}{dx} \right)^n G_0(x) \right]_{x=1}. \quad (7-5)$$

An important property of the generating functions is the so called “powers property:” if we choose m vertices at random from a large graph, then the distribution of the sum of the degrees of those vertices is generated by $[G_0(x)]^m$.

Another quantity of interest is the generating function of the distribution of the degree of the vertices that we arrive at by following a randomly chosen vertex:

$$G_1(x) = \frac{G'_0(x)}{G'_0(1)} = \frac{1}{z} G'_0(x). \quad (7-6)$$

Eq. (7-6) and the powers property allows to define the probability distribution of the second neighbors of a vertex as

$$\sum_k p_k [G_1(x)]^k = G_0(G_1(x)). \quad (7-7)$$

Consequently, the average number z_2 of second neighbors is

$$z_2 = \left[\frac{d}{dx} G_0(G_1(x)) \right]_{x=1} = G'_0(1)G'_1(1) = G''_0(1). \quad (7-8)$$

Now, let us consider the distribution of the sizes of connected components in the graph. Let $H_1(x)$ be the generating function for the distribution of the sizes of components which are reached by choosing a random edge and following it to one of its ends. It can be written as

$$H_1(x) = xG_1(H_1(x)). \quad (7-9)$$

If we start at a randomly chosen vertex, then we have one such component at the end of each edge leaving that vertex, and hence the generating function for the size of the whole component is

$$H_0(x) = xG_0(H_1(x)). \quad (7-10)$$

In principle, therefore, given the functions $G_0(x)$ and $G_1(x)$, we can solve Eq. (7-9) for $H_1(x)$ and substitute into Eq. (7-10) to get $H_0(x)$. Then we can find the probability that a randomly chosen vertex belongs to a component of size s by taking the s -th derivative of H_0 . In practice, unfortunately, this is usually impossible. However, we can find closed-form expressions for the average properties of clusters. For example, the average size of the component to which a randomly chosen vertex belongs, for the case where there is no giant component in the graph, after some manipulations, is given by

$$\langle s \rangle = 1 + \frac{G'_0(1)}{1 - G'_1(1)} = 1 + \frac{z_1^2}{z_1 - z_2}, \quad (7-11)$$

where $z_1 = z$ is the average number of first neighbors. We see that this expression diverges when $G'1(1) = 1$. This point marks the point at which a giant component first appears. Namely, from Eq. (7-11) we can derive the condition for the existence of a Giant Connected Component for undirected graphs as

$$\sum_k k(k-2)p_k > 0. \quad (7-12)$$

This result has been derived for the first time by Molloy and Reed in [91].

By definition, $H_0(x)$ generates the probability distribution of the sizes of components excluding the giant component. This means that $H_0(1)$ is no longer unity, as it is for the other generating functions considered so far, but instead takes the value $1 - S$, where S is the fraction of the graph occupied by the giant component. We can use this to calculate the size of the giant component from Eqs. (7-10) and (7-9) as

$$S = 1 - G_0(H_1(1)). \quad (7-13)$$

Thus, by leveraging these functions, we can derive lots of information starting by the only knowledge of the degree distribution. These relations have been exactly derived, for instance, for undirected random and scale-free networks, which exhibit a Poisson and a power-law distribution, respectively. For further details, see [96].

7.2.2 Generating Functions of Directed Networks

Some results regarding the generating functions have also been proposed for directed networks. In the following, we summarize the ones of our interest, taken from [96, 42]. The results are obtained for graphs with statistically uncorrelated vertices and an arbitrary joint in and out-degree

distribution $p(k_i, k_o)$, where k_i and k_o correspond to the in- and out-degree, respectively, and thus $k = k_i + k_o$ is the number of total connection of a node. In the view of this, the generating function of a directed network is

$$\Phi(x, y) = \sum_{k_i, k_o} p(k_i, k_o) x^{k_i} y^{k_o}, \quad (7-14)$$

with $|x|$ and $|y| \leq 1$. When the links are all inside the network, the average number of in- and out-degree is the same, thus being

$$\partial_x \Phi(x, 1) \Big|_{x=1} = \partial_y \Phi(1, y) \Big|_{y=1} \equiv z^{(d)}. \quad (7-15)$$

Therefore, the average degree is $z = 2z^{(d)}$.

If one ignores the directedness of edges, then the generating function becomes

$$\Phi^{(w)} = \Phi(x, x), \quad (7-16)$$

and

$$\Phi_1^{(w)}(x) = \frac{\Phi'^{(w)}(x)}{z}. \quad (7-17)$$

The criterion of Molloy and Reed for undirected networks (see Eq. (7-12)) can be used to check the existence in a digraph of a Giant Weakly Connected Component (GWCC, see Section 2.1.3.1), while the size of the GWCC can be computed as

$$W = 1 - \Phi^{(w)}(t_c), \quad t_c = \Phi_1^{(w)}(t_c). \quad (7-18)$$

In a directed network, we can also analyze the giant in- and out- component, GIN and GOUT, respectively (see Section 2.1.3.1 for their definition). Let us define the generating function of the out (in)-degree distribution of the vertex, approachable by following a randomly chosen edge moving *along* (*against*) the edge direction as

$$\Phi_1^{(o)}(y) = \frac{1}{z^{(d)}} \partial_x \Phi(x, y) \Big|_{x=1}; \quad \Phi_1^{(i)}(x) = \frac{1}{z^{(d)}} \partial_y \Phi(x, y) \Big|_{y=1}, \quad (7-19)$$

respectively.

In the view of this, the existence of the GIN and GOUT, and consequently of the GSCC, is ensured by

$$\Phi_1^{(i)}(1) = \Phi_1^{(o)}(1) = \frac{1}{z^{(d)}} \partial_{xy}^2 \Phi(x, y) \Big|_{x=1, y=1} > 1, \quad (7-20)$$

that is

$$\sum_{k_i, k_o} (2k_i k_o - k_i - k_o) p(k_i, k_o) > 0, \quad (7-21)$$

which corresponds to the criterion of Mollow and Reed for digraphs.

If Eq. (7-21) holds, then there are nontrivial solutions for the equations

$$x_c = \Phi_1^{(i)}(x_c) \quad y_c = \Phi_1^{(o)}(y_c). \quad (7-22)$$

$x_c < 1$ and $y_c < 1$ are the probabilities that the connected component reached by moving against (along) the edge directions, starting from a randomly chosen vertex, are finite. The in- and out- components of a vertex are sets of vertices approachable from this vertex moving against and along its edges, respectively, plus the vertex itself. Notice that any vertex that has only a finite out-component cannot, by definition, belong to the GIN.

Then, $p(k_i, k_o) x_c^{k_i}$ and $p(k_i, k_o) y_c^{k_o}$ are the probabilities that a random chosen edge with k_i incoming and k_o outgoing edge have finite in- and out- components. Summation of these expressions over (k_i, k_o) yields the total probability that the in- and out- components of a randomly chosen vertex are finite, that is, $\Phi(x_c, 1)$ and $\Phi(1, y_c)$, respectively.

From these considerations, we can derive the size of GIN and GOUT as

$$I = 1 - \Phi(x_c, 1), \quad O = 1 - \Phi(1, y_c), \quad (7-23)$$

respectively.

Notice that a vertex belongs to the GSCC if its in- and out- components are both infinite. Thus, the size of the GSCC in a digraph is

$$S = 1 - \Phi(x_c, 1) - \Phi(1, y_c) - \Phi(x_c, y_c), \quad (7-24)$$

where $\Phi(x_c, y_c)$ is the probability that both the in- and out- component of a vertex are finite. Eq. (7-24) can be explained in the following way. If at least one of a vertex's outgoing edges leads to anywhere in the GIN, then one can reach the GSCC from that vertex. Conversely, if at least one of a vertex's incoming edges leads from anywhere in the GOUT, then the vertex can be reached from the GSCC. If and only if both of these conditions are satisfied simultaneously, then the vertex belongs to the GSCC itself.

As for undirected networks, some results have been proposed for digraphs with well known degree distributions, see [92] for more details.

7.3 Generating Functions and Phase Transitions: Answered and Open Questions

The generating functions are actually not easy to manage. Moreover, they are based on some hypothesis that in general are difficult to satisfy. However, this approach presents different benefits, as it allows to describe lots of properties of a given network. Perhaps the real advantage of this approach is that it allows to deal with specific real-world graphs which have known degree distributions, known because we can measure them directly. For these graphs, one can measure the exact numbers n_k of vertices having degree k , and hence write down the exact generating function for that probability distribution in the form of a finite polynomial:

$$G_0(x) = \frac{\sum_k n_k x^k}{\sum_k n_k}, \quad (7-25)$$

where the sum in the denominator ensures that the generating function is properly normalized. Starting from Eq. (7-25), one can derive lots of information on a network which does not follow a specific degree distribution, as happens in almost all the real networks. The same, as obvious, could be done for a directed network.

For our purpose, this approach allows to give an answer to some of Questions (7.1.1). In particular, they may be reformulated in the following way:

Questions 7.3.1.

- 1) *Does a condition of the existence of the GSCC exist? If so, is it satisfied?*
- 2) *Are we able to compute the expected value of the size of the GSCC, stated its existence?*
- 3) *Can we predict the number of roots which enters the GSCC?*

Question 1) is easily to answer thanks to the aforementioned results. Namely, a condition for the existence of the GSCC actually exists for both undirected and directed networks. It corresponds to the criterion of Molloy and Reed in Eqs. (7-12) and (7-21). By checking this condition for a given network, one can predict which kind of phase transition will arise when a partial pinning control is applied, starting from the only knowledge of the degree distribution. Namely, if the condition is satisfied, we will observe a first order phase transition; otherwise, a second order phase transition will arise.

As to question 2), we have seen that, when a giant component is present, it is possible to estimate its size, see Eqs. (7-13) and (7-24). In this way, we can predict the number of nodes which could be pinning controlled if we would be able to pinning control the giant component.

Question 3) is perhaps the most interesting, but also the most difficult to answer. Namely, if we were able to predict the minimum number of nodes that must be pinned in order to pinning control the giant component, we could also evaluate a priori the convenience of such effort, in order to decide whether it's worth to spend resources for a particular aim. Actually, in order to predict p^* , one could predict the number of roots which enters the GSCC. Thus, we should have information on the nature of the nodes belonging to GIN. Unfortunately, although the generating functions' approach provides information on the size of the GIN (see Eq. (7-23)), up to now nothing more we can say about its composition.

Of course, this issue could be addressed in a future research. A possible way to answer this question could be to compute the probability that a node belongs to the GIN, given that its in-degree is equal to zero. In this way, we could have an estimate of the roots included in the GIN.

Conclusions

This thesis has been mostly motivated by the words of Jean-Claude Trichet, the former European Central Bank Governor, who, speaking about the recent financial crisis, highlighted the incapability of neoclassical economics of analyzing and predicting this sudden event. Thus, he encouraged the scientific community from other disciplines to provide tools that might complement traditional economics. In the view of this, we have tried to give our little contribution by exploiting our background in the field of control theory. Our interest focused on the phenomenon of informational cascades, which have been used as possible explanations for financial bubbles and crashes. However, we noticed that the classical models of informational cascades proposed in the literature are based on some restrictive assumptions which turned out to be unrealistic. For instance, they only consider the case of a total cascade, which is disconfirmed by empirical evidence. In the view of this, we tried to exploit tools from control theory to overcome some of these limitations. Namely, we proposed a novel model of opinion dynamics capable of triggering informational cascades of different intensities. This model allowed us to treat informational cascades as a pinning control

problem, and thus to leverage tools from partial pinning control to predict the emergence of such phenomena. We tested our opinion dynamic model in an agent-based model of artificial market, that we built for this purpose. We numerically showed how our model is capable of triggering informational cascades of different intensities. Moreover, we showed how these results confirmed the predictions made by leveraging tools from partial pinning control theory.

Eventually, we proposed a different approach which could allow to analytically characterize the partial pinning controllability of a given network with arbitrary degree distribution. Indeed, by inspecting the partial pinning controllability of the networks we have analyzed, we have found a close correlation with the topological features of the latter. In particular, the presence of a giant strongly connected component strongly affects the pinning controllability of the network. To better understand this correlation, we have tried to leverage tools from the generating functions' approach, which provides lots of information about the topological features of a network with given degree distribution, and thus could also allow to analytically predict information on the partial pinning controllability of complex networks. Indeed, thanks to this approach, we have already provided an answer on some issues on the topic. However, some criticisms are still unanswered, such as quantifying and predicting the effort that should be spent in order to pinning control most of a network, in order to evaluate the convenience of such effort. Given the novelty of the topic, it will be undoubtedly deeper investigated in the future.

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